Decentralised and distributed control

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Lucca
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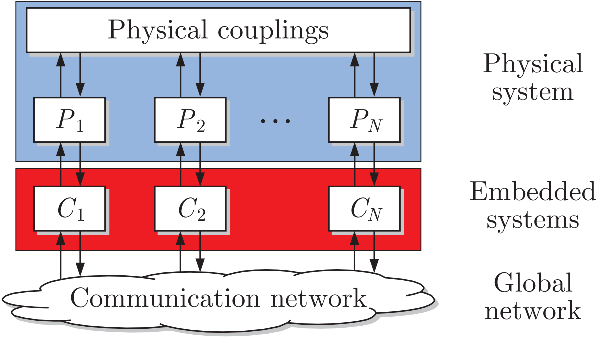
Decentralised and distributed control

Cyber-physical system:

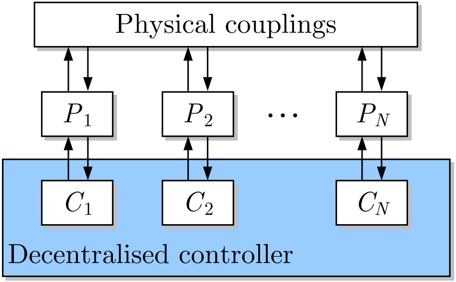


Which information is necessary to solve a control task?

- **Topology:** Which information links are necessary?
- **Quality:** Which accuracy of information is necessary?
- **Temporal aspect:** How quickly has information to be communicated?

Decentralised and distributed control

Decentralised control system:

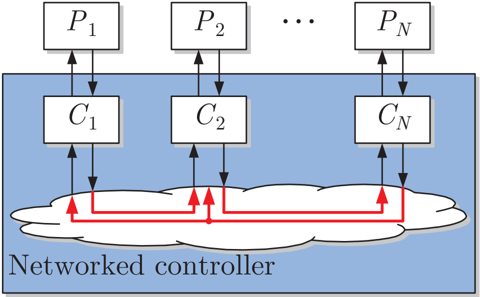


1. What are the consequences of structural constraints of the controller?
   → Decentralised stabilisability

2. How to design decentralised controllers?
   → Structurally constrained controllers

Decentralised and distributed control

Multi-agent system:



3. How to choose the information structure of distributed controllers?

Part I: Decentralised stabilisability

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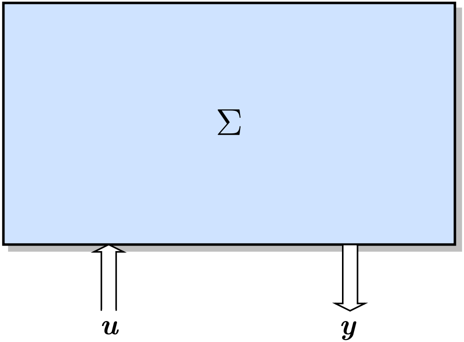
Part I:
Decentralised stabilisability

- Models of interconnected systems
- Decentralised and distributed controllers
- Decentralised fixed modes
- Structurally fixed modes

Models of interconnected systems

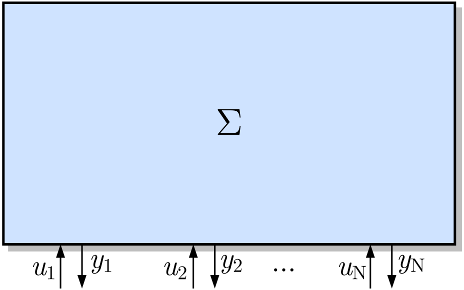
Models of interconnected systems



Unstructured model:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{cases}$$

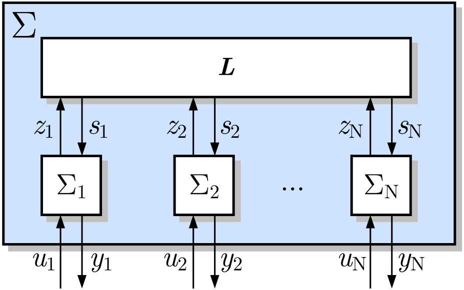
Models of interconnected systems



I/O-oriented model:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{N} \boldsymbol{b}_{\mathrm{s}i} u_i(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y_i(t) = \boldsymbol{c}_{\mathrm{s}i}^{\mathrm{T}} \boldsymbol{x}(t), \quad i \in \mathcal{N} = \{1, 2, ..., N\} \end{cases}$$

Models of interconnected systems



Interaction-oriented model:

$$\Sigma_i : \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}_i(t) & = & \boldsymbol{A}_i \boldsymbol{x}_i(t) + \boldsymbol{b}_i u_i(t) + \boldsymbol{e}_i s_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ z_i(t) & = & \boldsymbol{c}_{\mathrm{z}i}^{\mathrm{T}} \boldsymbol{x}_i(t) \\ y_i(t) & = & \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{x}_i(t) \end{array} \right.$$

Couplings: $\boldsymbol{s}(t) = \boldsymbol{L}\boldsymbol{z}(t)$

Models of interconnected systems

Relation between the interaction-oriented model and the unstructured model for

$$\boldsymbol{L} = \begin{pmatrix} 0 & l_{12} & \dots & l_{1N} \\ l_{21} & 0 & & l_{2N} \\ \vdots & & \ddots & \\ l_{N1} & l_{N2} & & 0 \end{pmatrix}$$

Unstructured model:

$$\Sigma : \left\{ \begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t) \end{array} \right.$$

with

$$\boldsymbol{B} = \mathrm{diag}\ \boldsymbol{b}_i, \quad \boldsymbol{C} = \mathrm{diag}\ \boldsymbol{c}_i^{\mathrm{T}}$$

Models of interconnected systems

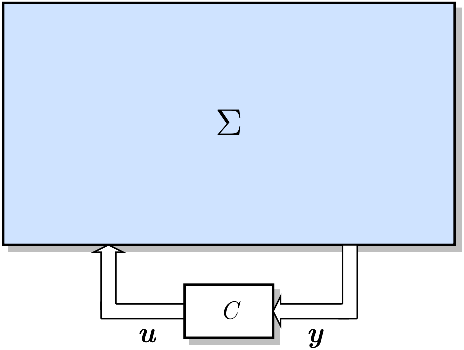
Relation between the interaction-oriented model and the unstructured model for

$$\boldsymbol{L} = \begin{pmatrix} 0 & l_{12} & \dots & l_{1N} \\ l_{21} & 0 & & l_{2N} \\ \vdots & & \ddots & \\ l_{N1} & l_{N2} & & 0 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{e}_1 l_{12} \boldsymbol{c}_{z2}^{\mathrm{T}} & \dots & \boldsymbol{e}_1 l_{1N} \boldsymbol{c}_{zN}^{\mathrm{T}} \\ \boldsymbol{e}_2 l_{21} \boldsymbol{c}_{z1}^{\mathrm{T}} & \boldsymbol{A}_2 & & \boldsymbol{e}_2 l_{2N} \boldsymbol{c}_{zN}^{\mathrm{T}} \\ \vdots & & \ddots & \\ \boldsymbol{e}_N l_{N1} \boldsymbol{c}_{z1}^{\mathrm{T}} & \boldsymbol{e}_N l_{N2} \boldsymbol{c}_{z2}^{\mathrm{T}} & & \boldsymbol{A}_N \end{pmatrix}$$

Decentralised and distributed controllers

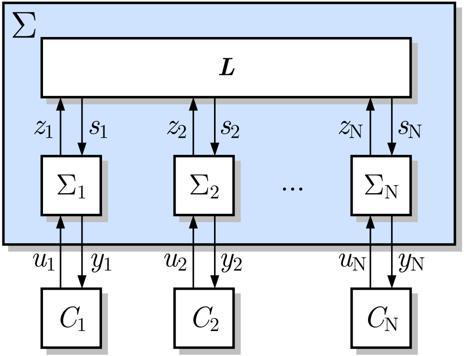
Decentralised and distributed controllers



Centralised controller:

$$C : \ \boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{y}(t)$$

Decentralised and distributed controllers



$$C: \quad \boldsymbol{u}(t) = -\begin{pmatrix} k_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & k_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & k_3 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & k_N \end{pmatrix} \boldsymbol{y}(t)$$

Decentralised and distributed controllers

Centralised controller: $$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1N} \\ k_{21} & k_{22} & \dots & k_{2N} \\ \vdots & \vdots & & \vdots \\ k_{N1} & k_{N2} & \dots & k_{NN} \end{pmatrix}$$

Decentralised controller: $$\boldsymbol{K} = \begin{pmatrix} k_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & k_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & k_3 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & k_N \end{pmatrix}$$

Distributed controller: $$\boldsymbol{K} = \begin{pmatrix} k_{11} & \mathbf{0} & k_{13} & \dots & \mathbf{0} \\ k_{21} & k_{22} & \mathbf{0} & & k_{2N} \\ \mathbf{0} & k_{32} & k_{33} & & \mathbf{0} \\ \vdots & & & \ddots & \\ k_{N1} & \mathbf{0} & k_{N3} & & k_{NN} \end{pmatrix}$$

$[\boldsymbol{K}]$ denotes the structure matrix of $\boldsymbol{K}$:

$$[\boldsymbol{K}] = \begin{pmatrix} * & \mathbf{0} & * & \dots & \mathbf{0} \\ * & * & \mathbf{0} & \dots & * \\ \mathbf{0} & * & * & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ * & \mathbf{0} & * & & * \end{pmatrix}$$

Structurally constrained controllers:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{y}(t)$$

with

$$\boldsymbol{K} \in \mathcal{K} = \{[\boldsymbol{K}] = \underset{\substack{\uparrow \\ \text{given structure matrix}}}{\boldsymbol{S}_{\mathrm{K}}}\}$$

Decentralised fixed modes

Decentralised fixed modes

I/O-oriented plant model:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{N} \boldsymbol{b}_{\mathrm{s}i} u_i(t) \\ y_i(t) & = & \boldsymbol{c}_{\mathrm{s}i}^{\mathrm{T}} \boldsymbol{x}(t), \quad i \in \mathcal{N} \end{cases}$$

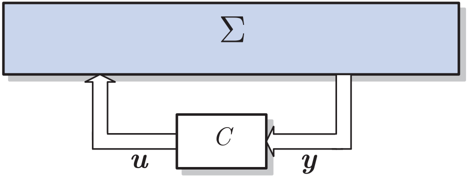
Decentralised controller:

$$u_i(t) = -k_i y_i(t)$$

Problem:
Under what conditions do feedback gains k_i, $(i \in \mathcal{N})$ exist such that the closed-loop system is asymptotically stable?

$$\bar{\Sigma} : \begin{cases} \dot{\boldsymbol{x}}(t) & = & (\boldsymbol{A} - \boldsymbol{B} \cdot \mathrm{diag}\,(k_i) \cdot \boldsymbol{C})\boldsymbol{x}(t) \\ y_i(t) & = & \boldsymbol{c}_{\mathrm{s}i}^{\mathrm{T}} \boldsymbol{x}(t), \quad i \in \mathcal{N} \end{cases}$$

Decentralised fixed modes



Answer for centralised controller: $\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{y}(t)$

An eigenvalue $\lambda \in \sigma(\boldsymbol{A})$ is not controllable and observable
$\Leftrightarrow$ It cannot be moved by static output feedback $\boldsymbol{K}$
$\Leftrightarrow$ It cannot be placed by dynamic output feedback $\boldsymbol{K}$

Decentralised fixed modes

Definition

The elements of the set

$$\Lambda = \bigcap_{\boldsymbol{K} \in \mathbb{R}^{N \times N}} \sigma(\boldsymbol{A} - \boldsymbol{BKC})$$

are said to be fixed modes (or fixed eigenvalues).

Λ is the set of uncontrollable or unobservable eigenvalues of $\boldsymbol{A}$:

$$\Lambda = \left\{ \lambda \,\middle|\, \text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{B}) < n \;\; \text{or} \;\; \text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{pmatrix} < n \right\}$$

Decentralised fixed modes

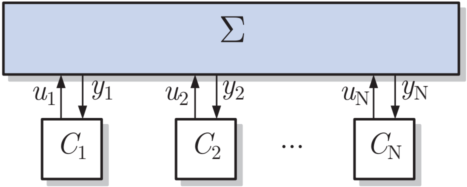
Consequences:

- If $\Lambda = \emptyset$: all eigenvalues can be moved by static feedback

  all eigenvalues can be placed by dynamic feedback

- If $\text{Re}(\lambda) < 0$ for all $\lambda \in \Lambda$, the plant is stabilisable.
- If $\exists \lambda \in \Lambda :\ \text{Re}(\lambda) \geq 0$, the plant is not stabilisable.

Decentralised fixed modes



What are the consequences of the structural constraints on the controller?

$$\boldsymbol{K} \in \mathcal{K} = \{\text{diag}\,(k_i) \,|\, k_i \in \mathbb{R},\ i \in \mathcal{N}\}$$

Decentralised fixed modes

Definition

The elements of

$$\Lambda_{\mathrm{d}} = \bigcap_{\boldsymbol{K} \in \mathcal{K}} \sigma(\boldsymbol{A} - \boldsymbol{BKC})$$

are said to be decentralised fixed modes (or decentralised fixed eigenvalues).

Restrict the consideration to static feedback.

Decentralised fixed modes

Under what conditions do decentralised fixed modes exist?

$$\Lambda_{\rm d} \neq \emptyset$$

Theorem

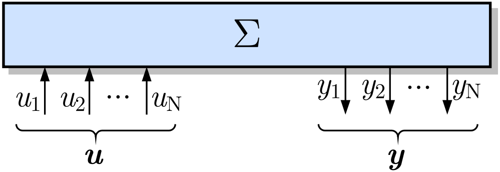
$\lambda \in \sigma(\boldsymbol{A})$ *is a decentralised fixed mode of* Σ

$\Longleftrightarrow$

$$\exists \mathcal{D}, \mathcal{H}: \quad \text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}_{\rm D} \\ \boldsymbol{C}_{\rm H} & \boldsymbol{O} \end{pmatrix} < n.$$

Why?

Decentralised fixed modes

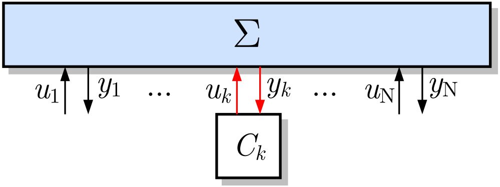


Preliminary results:

- As $\mathcal{K} \subseteq \mathbb{R}^{N \times N}$: $\Lambda_{\rm d} \supseteq \Lambda$.

  → Assumption:
  Σ is completely controllable and completely observable.

Decentralised fixed modes

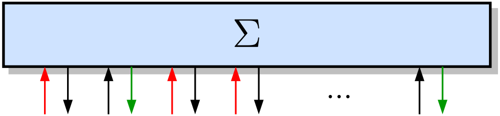


Preliminary results:

- If Σ is controllable and observable through a single channel (u_k, y_k), no decentralised fixed modes exist:

$$\left.\begin{array}{l} \mathrm{rank}(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{b}_{\mathrm{s}k}) = n \\ \mathrm{rank}\begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{c}_{\mathrm{s}k}^{\mathrm{T}} \end{pmatrix} = n \end{array} \quad \forall \lambda \in \sigma(\boldsymbol{A}) \right\} \Longrightarrow \quad \Lambda_{\mathrm{d}} = \emptyset.$$

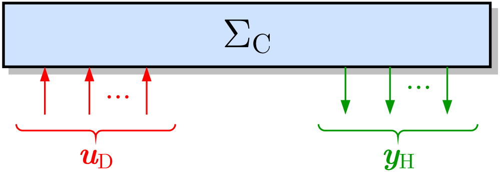
Decentralised fixed modes



$\lambda \in \sigma(\boldsymbol{A})$ can be a decentralised fixed mode only if it is not simultaneously controllable and observable through any channel (u_i, y_i)

$$\forall i \in \mathcal{N} : \begin{cases} \text{Either} & \operatorname{rank}(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{b}_{\mathrm{s}i}) < n \\ \text{or} & \operatorname{rank}\begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{c}_{\mathrm{s}i}^{\mathrm{T}} \end{pmatrix} < n \end{cases}$$

Decentralised fixed modes



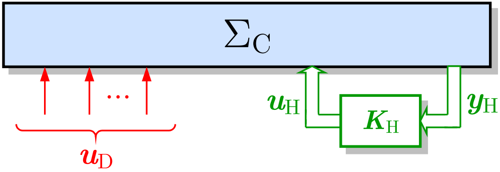
$$\text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{B}_{\text{D}}) \quad < \quad n$$

$$\text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C}_{\text{H}} \end{pmatrix} \quad < \quad n$$

Complementary system:

$$\Sigma_{\text{C}} : \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_{\text{D}}\boldsymbol{u}_{\text{D}}(t) \\ \boldsymbol{y}_{\text{H}}(t) &= \boldsymbol{C}_{\text{H}}\boldsymbol{x}(t) \end{cases}$$

Decentralised fixed modes



λ cannot be made controllable through $u_D(t)$ if

$$\text{rank}\,(\lambda I - A + B_H K_H C_H \;\; B_D) < n \quad \text{for all } K_H$$

$$(\lambda I - A + B_H K_H C_H \;\; B_D) =$$

$$(I \;\; O) \begin{pmatrix} I & B_H K_H \\ O & I \end{pmatrix} \begin{pmatrix} \lambda I - A & B_D \\ C_H & O \end{pmatrix}$$

Decentralised fixed modes

Theorem (Anderson, Clements, 1981)

$\lambda \in \sigma(\boldsymbol{A})$ *is a decentralised fixed mode of* Σ

$\Longleftrightarrow$

$$\exists \mathcal{D}, \mathcal{H}: \quad \text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}_{\text{D}} \\ \boldsymbol{C}_{\text{H}} & \boldsymbol{O} \end{pmatrix} < n.$$

$$\Lambda_{\text{d}} = \left\{ \lambda \,\middle|\, \exists \mathcal{D}, \mathcal{H}: \quad \text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}_{\text{D}} \\ \boldsymbol{C}_{\text{H}} & \boldsymbol{O} \end{pmatrix} < n \right\}$$

for centralised control:

$$\Lambda = \left\{ \lambda \,\middle|\, \text{rank}(\lambda \boldsymbol{I} - \boldsymbol{A} \;\; \boldsymbol{B}) < n \;\text{ or }\; \text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} \\ \boldsymbol{C} \end{pmatrix} < n \right\}$$

Decentralised fixed modes

Example:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{x}(t) \end{cases}$$

Σ is completely controllable and completely observable.

Decentralised fixed modes
Example

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{x}(t) \end{cases}$$

For $\lambda = -2$, $\mathcal{D} = \{1\}$ and $\mathcal{H} = \{2\}$:

$$\text{rank} \begin{pmatrix} \lambda \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{b}_{s1} \\ \boldsymbol{c}_{s2}^{T} & 0 \end{pmatrix} = \text{rank} \left(\begin{array}{ccc|c} -1 & -2 & -3 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array} \right) = 2 < 3$$

Hence, $\lambda = -2$ is a decentralised fixed eigenvalue.

Structurally fixed modes

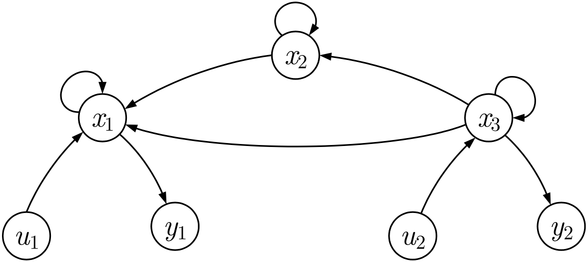
Structurally fixed modes

Aim:

Can we derive conditions on the structure of the system for the existence of fixed modes?

Structurally fixed modes

System structure:



Structure matrices:

$$[\boldsymbol{A}] = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad [\boldsymbol{B}] = \begin{pmatrix} * & 0 \\ 0 & 0 \\ 0 & * \end{pmatrix}, \quad [\boldsymbol{C}] = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$$

Structurally fixed modes

For given structure matrices $\boldsymbol{S}_{\mathrm{A}}$, $\boldsymbol{S}_{\mathrm{B}}$, $\boldsymbol{S}_{\mathrm{C}}$, consider the class of systems

$$\mathcal{S}(\boldsymbol{S}_{\mathrm{A}}, \boldsymbol{S}_{\mathrm{B}}, \boldsymbol{S}_{\mathrm{C}}) = \{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \,|\, [\boldsymbol{A}] = \boldsymbol{S}_{\mathrm{A}}, [\boldsymbol{B}] = \boldsymbol{S}_{\mathrm{B}}, [\boldsymbol{C}] = \boldsymbol{S}_{\mathrm{C}}\}$$

Definition

$\mathcal{S}$ is said to have structurally fixed modes if every system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathcal{S}$ has fixed modes:

$$\Lambda_{\mathrm{d}} = \bigcap_{\boldsymbol{K} \in \mathcal{K}} \sigma(\boldsymbol{A} - \boldsymbol{BKC}) \neq \emptyset.$$

Structurally fixed modes

Centralised control:

$\mathcal{K} = \mathrm{I\!R}^{N \times N}$:

$\mathcal{S}$ has structurally fixed modes

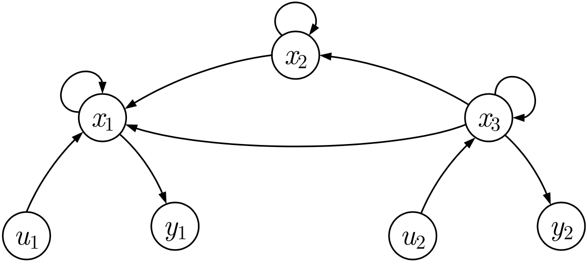
$\Leftrightarrow$

No system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}) \in \mathcal{S}$ is controllable and observable.

$\Leftrightarrow$

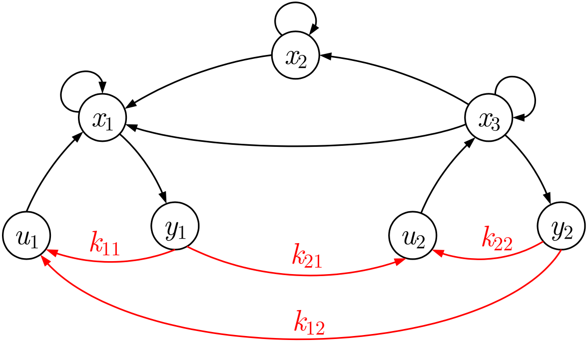
$\mathcal{S}$ is not structurally controllable or not structurally observable.

Structurally fixed modes



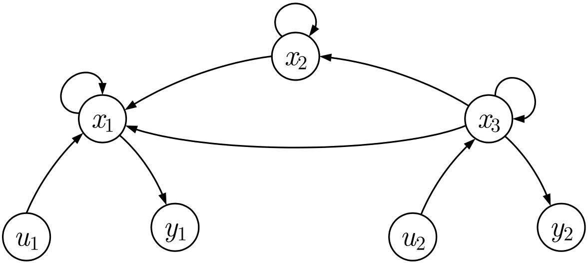
Structure graph of $\mathcal{S}$ (plant)

Structurally fixed modes



Structure graph of $\bar{S}$ (closed-loop system)

Structurally fixed modes



Theorem

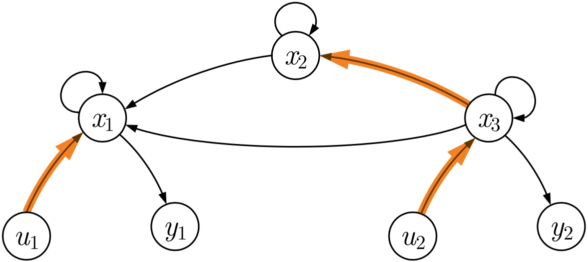
$\mathcal{S}$ *has structurally fixed modes for* $\mathcal{K} = \mathrm{IR}^{N \times N}$

$\Leftrightarrow$

at least one of the following conditions is satisfied:

- $\mathcal{S}$ *is either not input connectable or not output connectable.*
- *For* $\bar{\mathcal{S}}$ *there does not exist a cycle family of width* n.

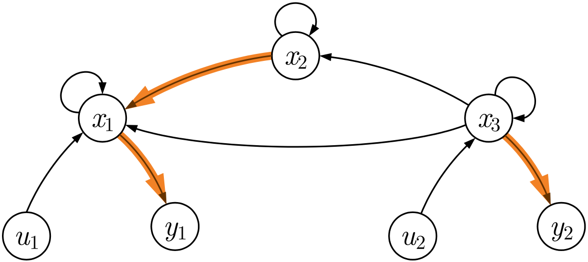
Structurally fixed modes



- $\mathcal{S}$ is input connectable (input reachable):

  There exist paths from the inputs u_i towards all state variables x_i

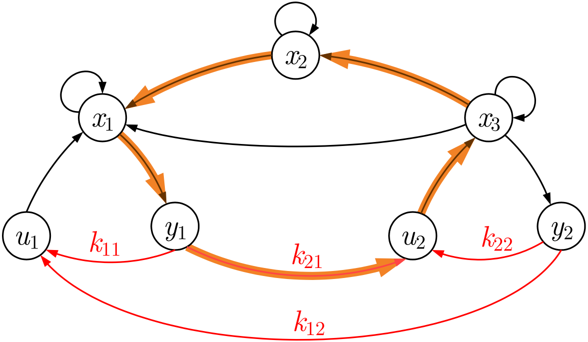
Structurally fixed modes



- $\mathcal{S}$ is output connectable (output reachable):

  There exist paths from all state variables x_i to at least one output y_i

Structurally fixed modes

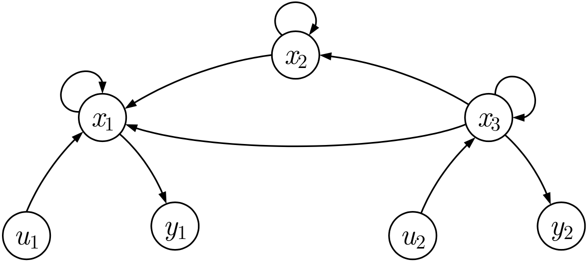


- $\bar{\mathcal{S}}$ has a cycle family of width n

  Cycle family = set of cycles without common vertex

  Width of cycle family = number of state vertices included

Structurally fixed modes



Result: $\mathcal{S}$ does not have structurally fixed modes for centralised control.

Structurally fixed modes

Extension to structurally constrained controllers:

Theorem

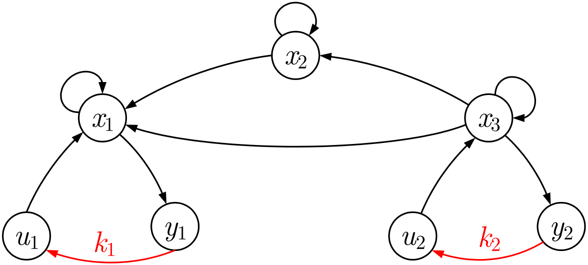
$\mathcal{S}$ has structurally fixed modes

$\Leftrightarrow$

at least one of the following conditions is satisfied:

- *In $\mathcal{S}$, there exists a state vertex that is not connectable to a channel.*
- *In $\bar{\mathcal{S}}$, there does not exist a cycle family of width n.*

Structurally fixed modes

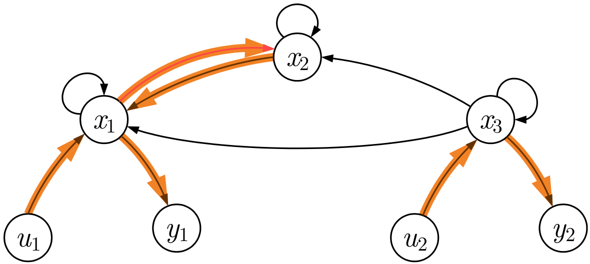


- x_2 is not connectable to a channel

Hence, $\mathcal{S}$ has structurally fixed modes for decentralised control

Structurally fixed modes

... with new edge:

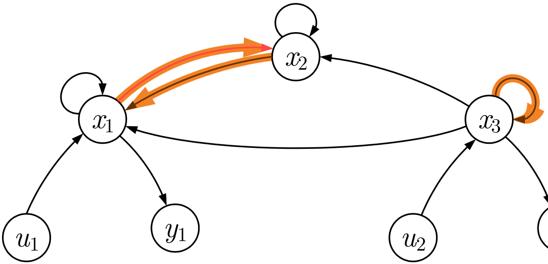


- x_2 is now connectable to the channel (u_1, y_1)
- There exists a cycle family of width $n = 3$.

Hence, no structurally fixed modes exist.

Structurally fixed modes

... with new edge:



- There exists a cycle family of width $n = 3$.

Hence, no structurally fixed modes exist.

Summary

Summary

Structural constraints on the controller make it „more difficult“ to control a systems:

- A plant may possess fixed modes with respect to the structural constraints, although it is controllable and observable by centralised control.
- Structural conditions show the consequences of structural constraints in $\mathcal{K}$

Summary

Structural conditions show the consequences of structural constraints in $\mathcal{K}$:

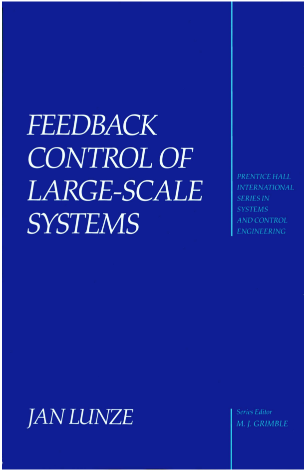
- The state vertices have to be connectable to a channel.
- The cycle family has to exist in the graph $\bar{\mathcal{S}}$, which depends upon $\mathcal{K}$.

Structurally fixed modes exist $\Rightarrow$ Fixed modes exist

$\not\Leftarrow$

Fixed modes may exist for specific parameters

Literature (1):



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- Trave, L.; Titli, A.; Tarras, A. M.:
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- Sezer, M. E.; Siljak, D. D.: Structurally fixed modes, *Syst. Control Letters* **1** (1981), pp. 60-64.

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Part II: Design of decentralised controllers

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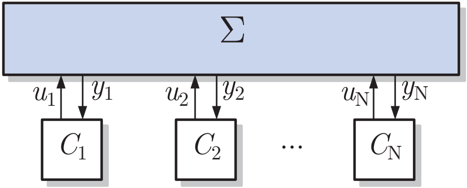
Part II:
Design of decentralised controllers

- Optimal decentralised control
- Direct Nyquist Array Method for designing decentralised controllers
- Example: Decentralised voltage control of an electric power system
- Extensions
- Summary

Optimal decentralised control

Optimal decentralised control



I/O-oriented plant model:

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{N} \boldsymbol{b}_{\mathrm{s}i} u_i(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y_i(t) = \boldsymbol{c}_{\mathrm{s}i}^{\mathrm{T}} \boldsymbol{x}(t), \quad i \in \mathcal{N} \end{cases}$$

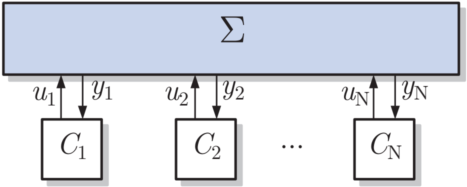
Find a decentralised controller

$$C_i : \quad u_i(t) = -k_i y_i(t)$$

such that $J = \int_0^{\infty} \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{u}(t)\,\mathrm{d}t \to \min\limits_{k_1,\dots,k_N}$

Optimal decentralised control

Optimal decentralised feedback:



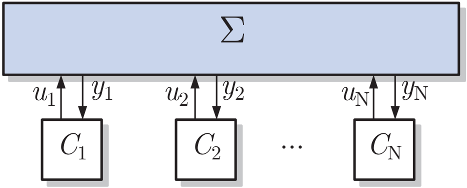
$$\min_{\boldsymbol{K} \in \mathcal{K}} \bar{J} \quad \text{with} \quad \bar{J} = \text{tr}\, \boldsymbol{P}$$

and

$$(\boldsymbol{A} - \boldsymbol{BKC})^{\mathrm{T}} \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{BKC}) + \boldsymbol{C}^{\mathrm{T}} \boldsymbol{K}^{\mathrm{T}} \boldsymbol{RKC} + \boldsymbol{Q} = \boldsymbol{O}$$

Optimal decentralised control

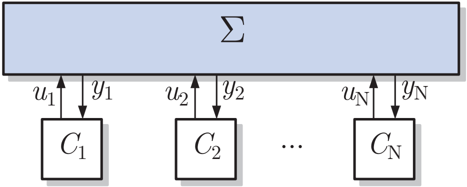
Optimal decentralised feedback:



Necessary optimality condition:

$$\frac{\mathrm{d}\bar{J}}{\mathrm{d}\boldsymbol{K}} = 0$$

Optimal decentralised feedback:



Necessary optimality conditions: (Levine, Athans, 1970)

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{L}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{C}\boldsymbol{L}\boldsymbol{C}^{\mathrm{T}})^{-1}$$

$$\boldsymbol{O} = (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}\boldsymbol{C})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}\boldsymbol{C}) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K}\boldsymbol{C} + \boldsymbol{Q}$$

$$\boldsymbol{O} = (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}\boldsymbol{C})\boldsymbol{L} + \boldsymbol{L}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}\boldsymbol{C})^{\mathrm{T}} + \boldsymbol{I}$$

Optimal decentralised control

Iterative solution: for $u_i(t) = -\boldsymbol{k}_i^{\mathrm{T}} \boldsymbol{x}_i(t)$

Given: stabilising decentralised feedback $\boldsymbol{K}^0 = \mathrm{diag}\, \boldsymbol{k}_i^{\mathrm{T}0}$

$k = 0$

1. Find $\boldsymbol{P}^{k+1}$ and $\boldsymbol{L}^{k+1}$ for $\boldsymbol{K}^k$
2. Determine $\frac{\mathrm{d}\bar{J}}{\mathrm{d}\boldsymbol{K}} = 2(\boldsymbol{R}\boldsymbol{K}^k\boldsymbol{C} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}^{k+1})\boldsymbol{L}^{k+1}\boldsymbol{C}^{\mathrm{T}}$
3. Put the diagonal elements of $\frac{\mathrm{d}\bar{J}}{\mathrm{d}\boldsymbol{K}}$ into the matrix $\boldsymbol{D}^{k+1}$
4. Determine a step size s^{k+1} such that

$$\bar{J}(\underbrace{\boldsymbol{K}^k - s^{k+1}\boldsymbol{D}^{k+1}}_{=\boldsymbol{K}^{k+1}}) < \bar{J}(\boldsymbol{K}^k)$$

5. If $\|\boldsymbol{D}^{k+1}\| < \epsilon$, stop;
   otherwise $k = k + 1$, repeat from Step 1.

Optimal decentralised control

Properties of this iterative solution:
(Geromel, Bernussou, 1979)

- In each step, the performance $\bar{J}$ is improved.
- If initialised with a stabilising feedback $\boldsymbol{K}^0$, each controller $\boldsymbol{K}^k$ is a stabilising decentralised controller.

Optimal decentralised control

Evaluation:

- For the initialisation of the algorithm, a stabilising decentralised feedback $\boldsymbol{K}^0$ has to be found.
- This is a centralised design method.

Better design methods:

a) Hierarchical design methods (cf. hierarchical optimisation)

b) Decentralised design methods

Direct Nyquist Array Method for designing decentralised controllers

Direct Nyquist Array Method

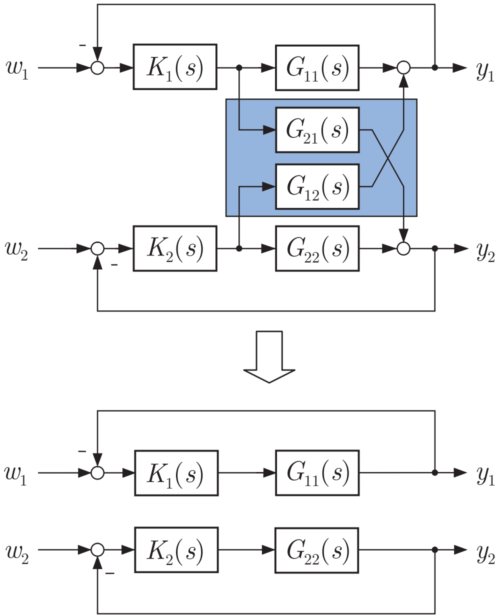
Idea:

Consider weakly coupled subsystems.

Design the control stations for the isolated subsystems.

Check conditions under which this decentralised controller „works"?

Direct Nyquist Array Method



Direct Nyquist Array Method

$$\Sigma : \qquad \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

$$C_i : \qquad U_i(s) = -K_i(s)(Y_i(s) - W_i(s)), \quad i = 1, 2.$$

Design the control stations $K_i(s)$ separately for the isolated subsystems

$$\Sigma_i : \quad Y_i(s) = G_{ii}(s)U_i(s)$$

so as to satisfy local requirements.

Direct Nyquist Array Method

Closed-loop system with decentralised controller:

$$\bar{\Sigma}: \quad \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \boldsymbol{G}_0(s)(\boldsymbol{I} - \boldsymbol{G}_0(s))^{-1} \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}$$

with

$$\boldsymbol{G}_0(s) = \begin{pmatrix} K_1(s)G_{11}(s) & K_1(s)G_{12}(s) \\ K_2(s)G_{21}(s) & K_2(s)G_{22}(s) \end{pmatrix}$$

Under what conditions are the couplings $G_{12}(s)$, $G_{21}(s)$ „weak“?

Direct Nyquist Array Method

Stability analysis of the overall system:
by means of the Nyquist criterion

$$\boldsymbol{F}(s) = \boldsymbol{I} + \boldsymbol{G}_0(s) = \begin{pmatrix} 1 + K_1(s)G_{11}(s) & K_1(s)G_{12}(s) \\ K_2(s)G_{21}(s) & 1 + K_2(s)G_{22}(s) \end{pmatrix}$$

Assumptions:

- $\boldsymbol{G}(s)$ is asymptotically stable.
- All isolated closed-loop subsystems are asymptotically stable.

Direct Nyquist Array Method

Stability analysis of the overall system:
by means of the Nyquist criterion

$$\boldsymbol{F}(s) = \boldsymbol{I} + \boldsymbol{G}_0(s) = \begin{pmatrix} 1 + K_1(s)G_{11}(s) & K_1(s)G_{12}(s) \\ K_2(s)G_{21}(s) & 1 + K_2(s)G_{22}(s) \end{pmatrix}$$

Under what conditions can the cross-couplings $G_{21}(s), G_{12}(s)$ <u>not</u> change the encirclement of the origin by $\det \boldsymbol{F}(s)$ („weak couplings“)?

Reformulation of the Nyquist criterion:

As

$$\det \boldsymbol{F}(s) = \prod_{i=1}^{m} \lambda_{\mathrm{F}i}(s)$$

with

$$\det \left(\lambda_{\mathrm{F}i}(s)\boldsymbol{I} - \boldsymbol{F}(s)\right) = 0$$

we get

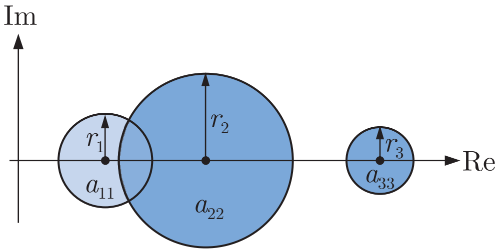
$$\Delta \arg \det \boldsymbol{F}(s) = \sum_{i=1}^{m} \Delta \arg \lambda_{\mathrm{F}i}(s)$$

Sufficient stability condition:

$$\begin{aligned} \Delta \arg \lambda_{\mathrm{F}i}(s) &= \Delta \arg(1 + K_i(s) G_{ii}(s)) \\ &= \Delta \arg F_{ii}(s), \quad i \in \mathcal{N} \end{aligned}$$

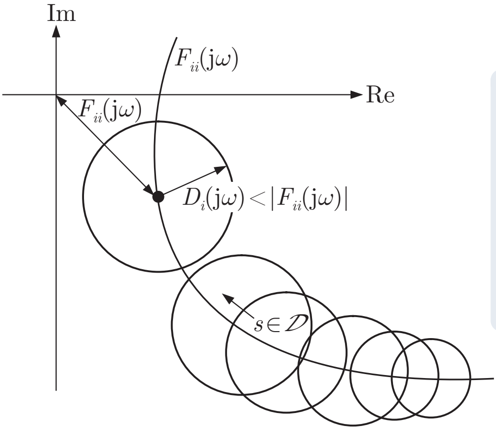
for

$$\boldsymbol{F}(s) = \begin{pmatrix} F_{11}(s) & F_{21}(s) \\ F_{12}(s) & F_{22}(s) \end{pmatrix}$$



GERSHGORIN'S Theorem:

$$|\lambda_{\mathrm{F}i}(s) - F_{ii}(s)| \leq \sum_{j=1, j\neq i}^{m} |F_{ij}(s)| = D_i(s)$$



The couplings $F_{ij}(s)$ do not change the encirclement of the origin if

$$|F_{ii}(s)| > \sum_{j=1, j\neq i}^{m} |F_{ij}(s)| \text{ for } s \in \mathcal{D}$$

Definition (Rosenbrock 1974)

$\boldsymbol{F}(s)$ is said to be diagonal dominant if

$$|F_{ii}(s)| > \sum_{j=1, j\neq i}^{m} |F_{ij}(s)| \quad \text{for} \quad s \in \mathcal{D}, \quad i \in \mathcal{N}$$

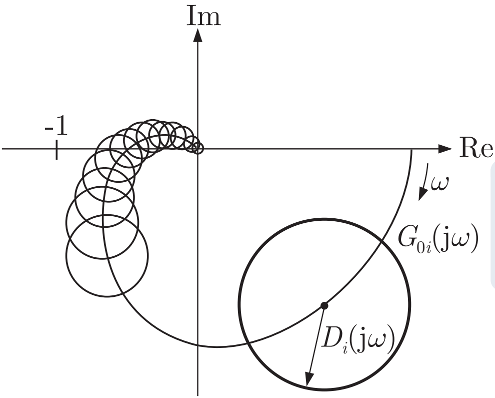
Theorem

$\boldsymbol{G}_0(s)$ – *stable*

$\Delta \arg(1 + K_i(s) G_{ii}(s)) = 0$

Then the decentralised control system is stable if $\boldsymbol{F}(s)$ *is diagonal dominant.*

Reformulation of the theorem for SISO subsystems:



If $\boldsymbol{F}(s)$ is diagonal dominant, the Gershgorin bands do not include the point -1

$$G_{0i}(j\omega) = K_i(j\omega)G_{ii}(j\omega)$$

Consequences:

- Diagonal dominance determines what the attribute „weak couplings“ mean.
- „Weakness“ of couplings depend upon the controllers $K_i(s)$ used.

$$\boldsymbol{F}(s) = \begin{pmatrix} 1 + K_1(s)G_{11}(s) & K_1(s)G_{12}(s) \\ K_2(s)G_{21}(s) & 1 + K_2(s)G_{22}(s) \end{pmatrix}$$

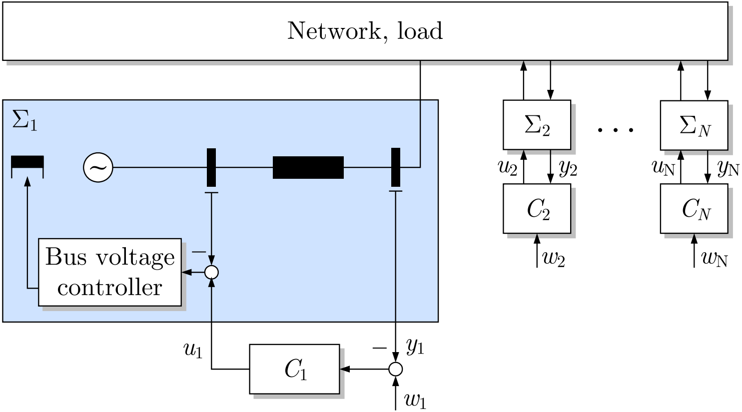
Extension:

If $\boldsymbol{F}(s)$ is diagonal dominant, stability is ensured even if subsystems are switched off (integrity).

$$\boldsymbol{F}(s) = \begin{pmatrix} 1+K_1(s)G_{11}(s) & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & 1+K_2(s)G_{22}(s) & K_2(s)G_{23}(s) \\ \boldsymbol{O} & K_3(s)G_{32}(s) & 1+K_3(s)G_{33}(s) \end{pmatrix}$$

Example:
Decentralised voltage control
of an electric power system

Decentralised voltage control



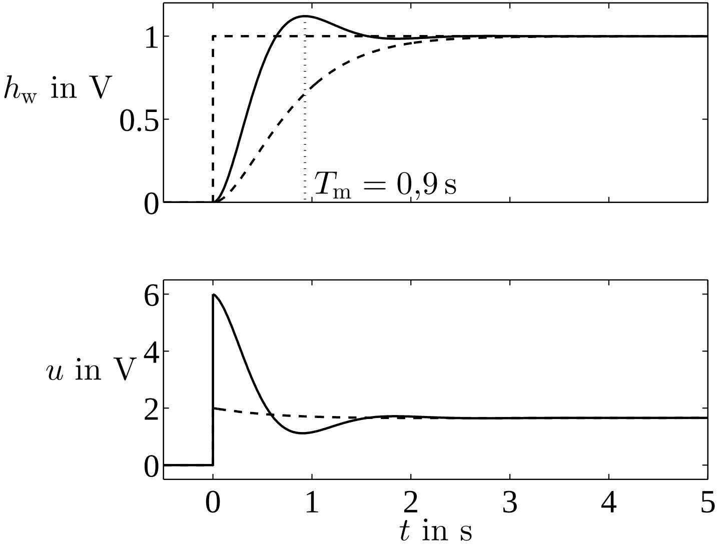
Decentralised voltage control

1. Design of the decentralised control stations:

$$U_i(s) = -K_i(s)(Y_i(s) - W_i(s))$$

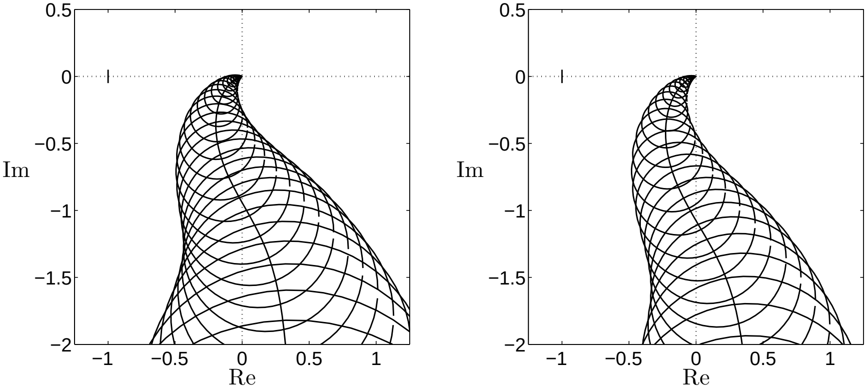
with

$$K_i(s) = \left(1 + \frac{2}{s}\right) \frac{1+s}{1+0.22s}$$



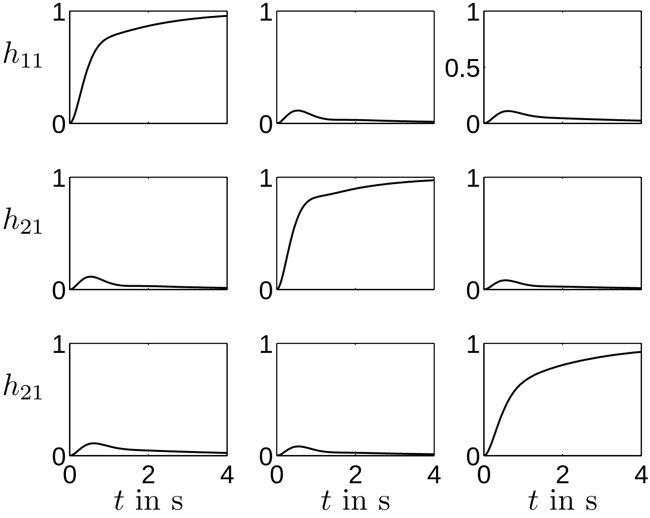
h_w in V
1
0.5
0
$T_m = 0{,}9\,s$
u in V
6
4
2
0
0
1
2
3
4
5
t in s

2. Analysis of the overall closed-loop system:



Hence, the overall system is stable.

Command step response matrix:



Decentralised voltage control

Summary of the Direct Nyquist Array Method:

1. *Design control stations $K_i(s)$ for the isolated subsystems Σ_i.*
2. *Evaluate the performance of the overall closed-loop system:*

   Diagonal dominance of $\boldsymbol{F}(s) \rightarrow$ stability, integrity

Evaluation:

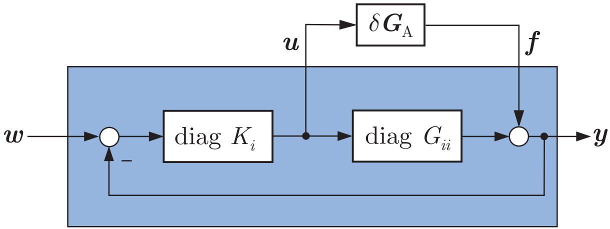
+ Decentralised design
- Conservative analysis

Extensions

Extensions

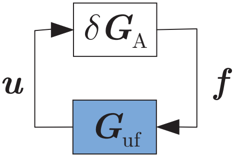
Evaluation of the I/O behaviour

Idea: Consider the couplings as model uncertainties $\delta\boldsymbol{G}_{\mathrm{A}}(s)$:



$$\delta\boldsymbol{G}_{\mathrm{A}}(s) = \begin{pmatrix} \boldsymbol{O} & G_{12}(s) & G_{13}(s) \\ G_{21}(s) & \boldsymbol{O} & G_{23}(s) \\ G_{31}(s) & G_{32}(s) & \boldsymbol{O} \end{pmatrix}$$

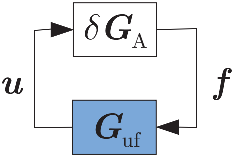
Extensions



$$\begin{aligned} \boldsymbol{G}_{\rm uf}(s) &= -\boldsymbol{K}(\boldsymbol{I} + {\rm diag}\, G_{ii}(s)\boldsymbol{K})^{-1} \\ &= -{\rm diag}\, \frac{K_i(s)}{1 + G_{ii}(s)K_i(s)} \end{aligned}$$

$$\delta\boldsymbol{G}_{\rm A}(s)\boldsymbol{G}_{\rm uf}(s) = {\rm diag}\, \frac{1}{F_{ii}(s)} \begin{pmatrix} 0 & F_{12}(s) & F_{13}(s) \\ F_{21}(s) & 0 & F_{23}(s) \\ F_{31}(s) & F_{32}(s) & 0 \end{pmatrix}$$

Extensions



Sufficient stability condition

$$\lambda_P \left\{ \text{diag}\, \frac{1}{|F_{ii}(s)|} \begin{pmatrix} 0 & |F_{12}(s)| & |F_{13}(s)| \\ |F_{21}(s)| & 0 & |F_{23}(s)| \\ |F_{31}(s)| & |F_{32}(s)| & 0 \end{pmatrix} \right\} < 1, \quad s \in \mathcal{D}$$

λ_P – Largest eigenvalue („Perron root“)

Extensions

$\boldsymbol{F}(s)$ is said to be generalised diagonal dominant, if

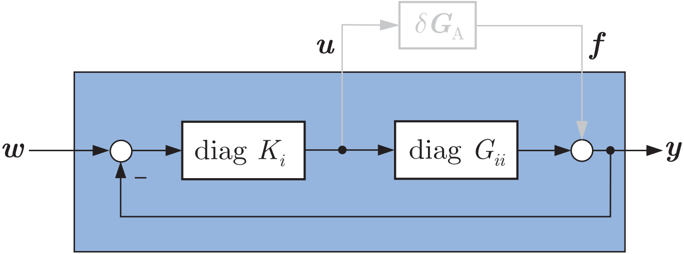
$$\lambda_{\rm P}\left\{\text{diag}\,\frac{1}{|F_{ii}(s)|}\begin{pmatrix} 0 & |F_{12}(s)| & |F_{13}(s)| \\ |F_{21}(s)| & 0 & |F_{23}(s)| \\ |F_{31}(s)| & |F_{32}(s)| & 0 \end{pmatrix}\right\} < 1,\ \ s \in \mathcal{D}$$

Hence, the overall system is stable if

- all isolated closed-loop subsystems are stable
- $\boldsymbol{F}(s)$ is generalised diagonal dominant

Extensions

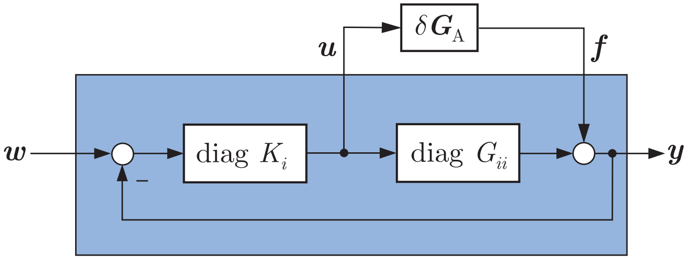
Evaluation of the I/O-behaviour:



Approximation by isolated subsystems:

$$\hat{Y}_i(s) = \frac{G_{ii}(s)K_i(s)}{1 + G_{ii}(s)K_i(s)} W_i(s)$$

Evaluation of the I/O-behaviour:



Approximation error bound

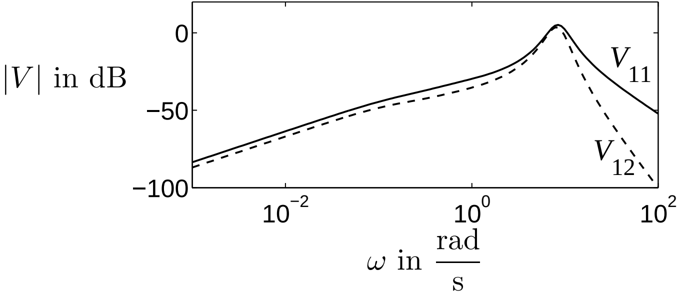
$$|\boldsymbol{Y}(s) - \hat{\boldsymbol{Y}}(s)| \leq \boldsymbol{V}(s)|\boldsymbol{W}(s)|$$

with

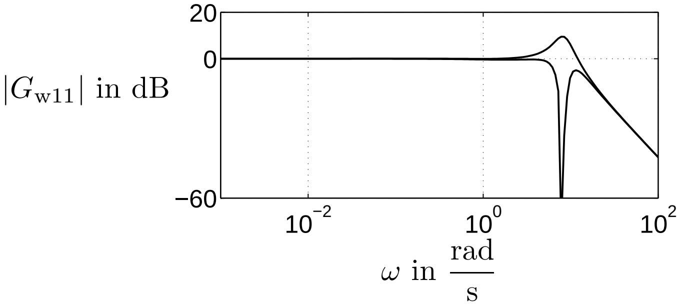
$$\boldsymbol{V}(s) = |\boldsymbol{G}_{\rm yf}(s)|\,|\delta\boldsymbol{G}_{\rm A}(s)|\,(\boldsymbol{I} - |\boldsymbol{G}_{\rm uf}(s)|\,|\delta\boldsymbol{G}_{\rm A}(s)|)^{-1}\,|\boldsymbol{G}_{\rm uw}(s)|$$

Change control laws to

$$K_i(s) = 6 \left(1 + \frac{2}{s}\right) \frac{1+s}{0.22\,s+1}.$$



Command behaviour



Summary

Summary

- Structural constraints of the controller make the design problem „difficult“.
- Centralised design methods (like optimal control) do not scale with the size of the plant.
- Decentralised design methods are useful for „weakly coupled“ interconnected systems.

Literature (1):

- Levine, W.; Athans, M.: On the determination of optimal output feedback gains for linear systems, *IEEE Trans. Autom. Control* **AC-15** (1970), pp. 44-49.
- Geromel, J. C.; Bernoussou, J.: An algorithm for optimal decentralized regulation of linear quadratic interconnected systems, *Automatica* **15** (1979), pp. 489-491.
- Rosenbrock, H. H.: *Computer-Aided Control Systems Design*, Academic Press, London 1974.
- Berman, A.; Plemmons, R. J.: *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York 1973.

Literature (2):

Lunze, J. :
Regelungstechnik, Band 2,
Springer-Verlag, Heidelberg 2012.

Part III: Design of the information structure of networked controllers

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July 3, 2013

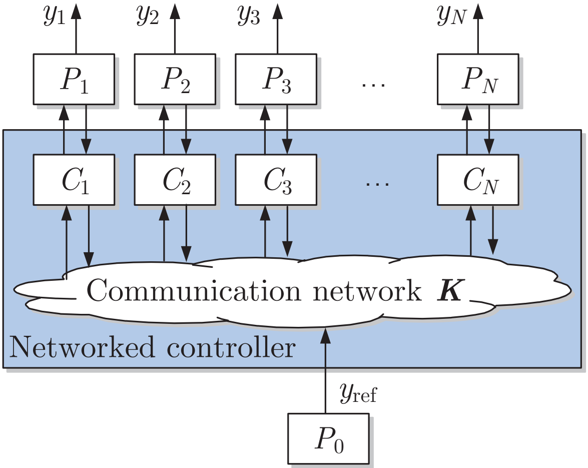
Part III:
Design of the information structure of networked controllers

- Information structure of networked controllers for multi-agent systems
- Results of Network Science
- Design method for the information structure of networked controllers
- Summary

Information structure
of networked controllers

Information structure of networked controllers

Leader-follower synchronisation



The overall system is said to be synchronised, if it satisfies the following requirements:

- **Synchronous behaviour**
  (for specific initial states $\boldsymbol{x}_{i0}, \boldsymbol{x}_{s0}$):

$$y_1(t) = y_2(t) = \ldots = y_N(t) = y_{\text{ref}}(t) \quad t \geq 0,$$

- **Asymptotic synchronisation**
  (for all other initial states):

$$\lim_{t\to\infty} |y_i(t) - y_{\text{ref}}(t)| = 0, \quad i = 1, 2, \ldots, N.$$

Literature:

- Pecora, L. M.; Carroll, T. L.: Master stability functions for synchronized coupled systems, *Physical Review Letters* **80** (1998)
- Olfati-Saber, R.; Fax, J. A.; Murray, R. M.: Consensus and cooperation in networked multi-agent systems, *Proc. of the IEEE* **95** (2007)
- Chen, G.; Duan, Z.: Network synchronizability analysis: A graph-theoretic approach, *Chaos* **18** (2008)
- Scardovi, L.; Sepulchre, R.: Synchronization in networks of identical linear systems, *Automatica* **45** (2009)

Summary of results in literature:

The majority of papers is restricted to synchronisation as an asymptotic property: $\lim_{t\to\infty} |y_i(t) - y_{\text{ref}}(t)| = 0.$

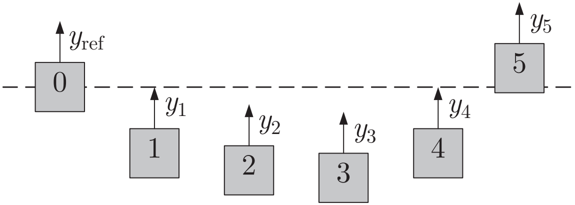
This talk gives an answer to the question:

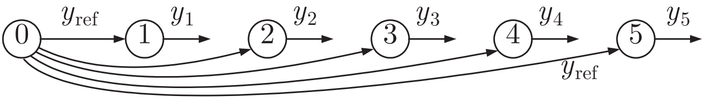
For which information structure does the overall system have a good transient behaviour?

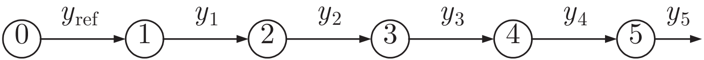
$$J_i = \int_0^\infty |y_{\text{ref}}(t) - y_i(t)|\, \mathrm{d}t, \quad i = 1, 2, , ..., N.$$

Information structure of networked controllers

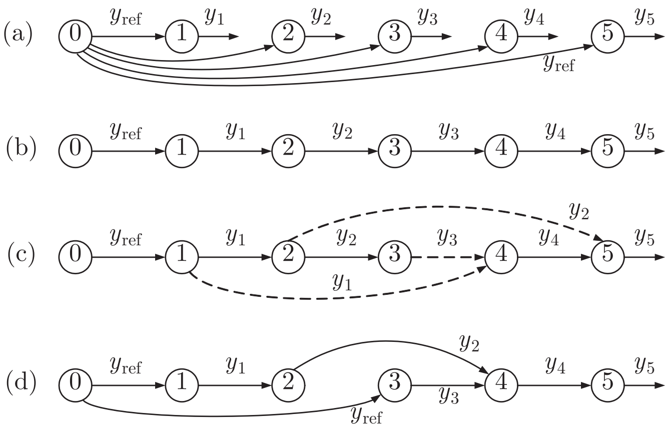
Example: Robot formation problem







Different information structures:



Which information structure leads to the best transient behaviour?

Information structure of networked controllers

New methodological questions for the design of networked controllers:

1. Which additional information is necessary to improve the performance of the closed-loop system?
2. How should additional information be utilised by the controller?
3. **Design of the information structure:**
   Which couplings among the controllers of the subsystems should be used?

Information structure of networked controllers

Two answers will be given in the following:

1. Network Science shows that existing large networks have a short characteristic path length.
2. A new structural design method will be proposed that uses a quantitative evaluation of the communicated information for the closed-loop system performance.

Results of Network Science

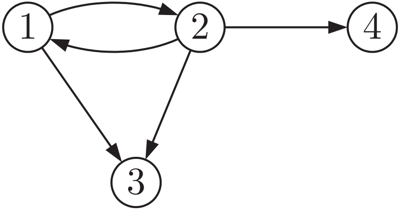
Results of Network Science

Network Science:
study of the network structure of existing large networks

- **Networks without hierarchy or coordinator:**
  Which structures appear as a result of self-organisation?
- **Network dynamics:**
  How do networks develop according to microlevel rules?

Results of Network Science

Some basics of graph theory:



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with

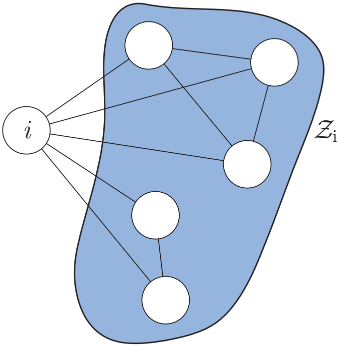
- $\mathcal{V}$ – set of nodes
- $\mathcal{E}$ – set of edges: $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

Characteristic path length:

$$\bar{l} = \frac{1}{N} \sum_{i,j \in \mathcal{V}} \min_{P(i,j) \in \mathcal{P}(i,j)} |P(i,j)|$$

with

- N – number of connected node pairs,
- $\mathcal{P}(i,j)$ – set of paths $P(i,j)$ from i towards j.



Cluster coefficient:

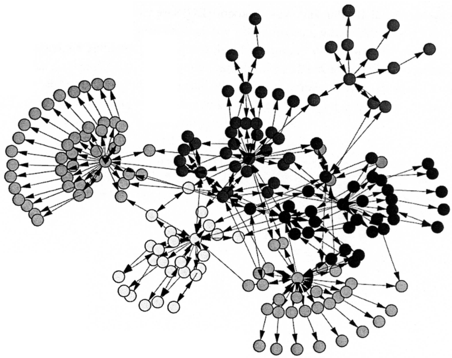
$$c_i = \frac{2e_i}{|\mathcal{Z}_i|(|\mathcal{Z}_i| - 1)}$$

with e_i = number of edges in $\mathcal{Z}_i$

(„How many friends of i are also friends among each other?“)

Results of Network Science

Three important properties of large existing networks:



Small-world architecture
(„How small the world is!")

- Short characteristic path length
- Large clustering coefficient

Scale-free networks

- Power-law degree distribution

Imitation of the dynamics of large existing networks:

Network = dynamic graph

$$\mathcal{G}(t) = (\mathcal{V}(t), \mathcal{E}(t), J)$$

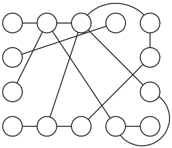
with

- J – microrules that govern the evolution of the graph

What are the microrules J that bring about such graphs?

Results of Network Science

Graphs between randomness and order



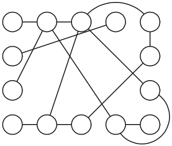
Random graph

Random graph: (Erdös, Renyi (1959))

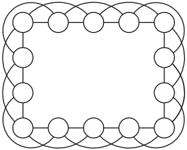
Each edge (i, j) exists with probability p.

- Small characteristic path length $\bar{l} \sim \log(N)$
- Small clustering coefficient $c = p$

Graphs between randomness and order



Random graph



Regular graph

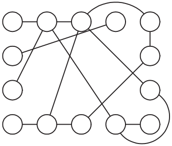
Regular graph:

Each node is connected with k neighbours.

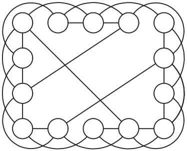
- Large characteristic path length $\bar{l} \sim N$
- Large clustering coefficient,
  e. g. $c \approx \frac{3}{4}$ for $k \gg \log(N)$

Results of Network Science

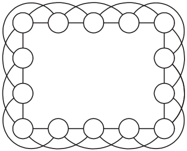
Graphs between randomness and order



Random graph



Small-world network



Regular graph

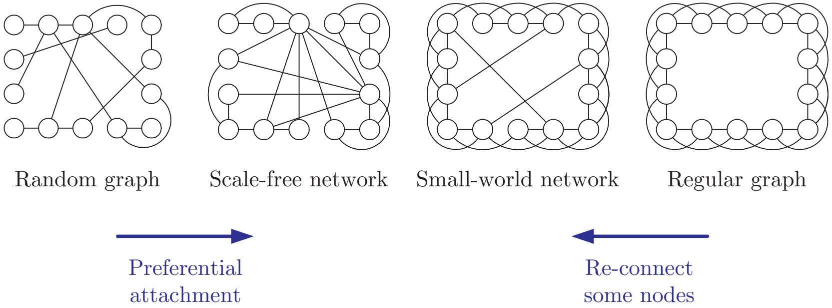
Re-connect
some nodes

Small-world network:

A few shortcuts reduce the characteristic path length considerably.

Results of Network Science

Graphs between randomness and order



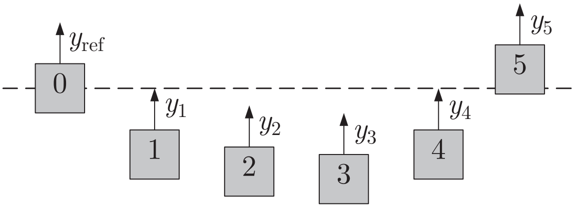
Scale-free networks:

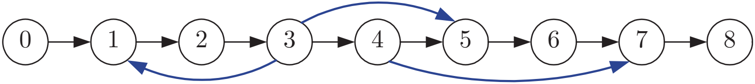
- Preferential attachment yields clusters
- Power-law degree distribution

Network dynamics:
How do networks develop according to microlevel rules?

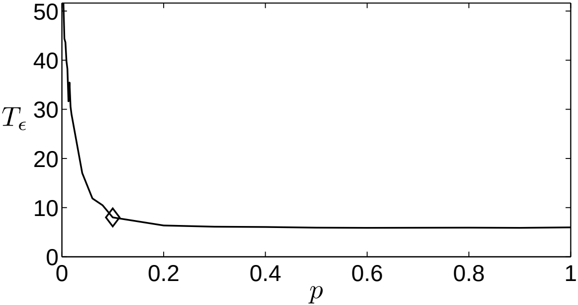
Phase transitions:
If the connectivity of the graph exceeds a critical value, new network properties abruptly appear.

Example: Robot formation problem

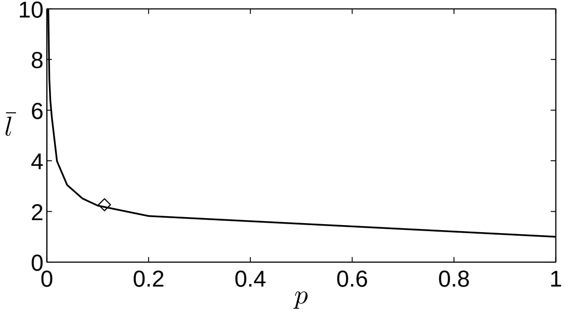




Transient behaviour with random additional communication



Graph-theoretic analysis of the result



Results of Network Science

Summary:

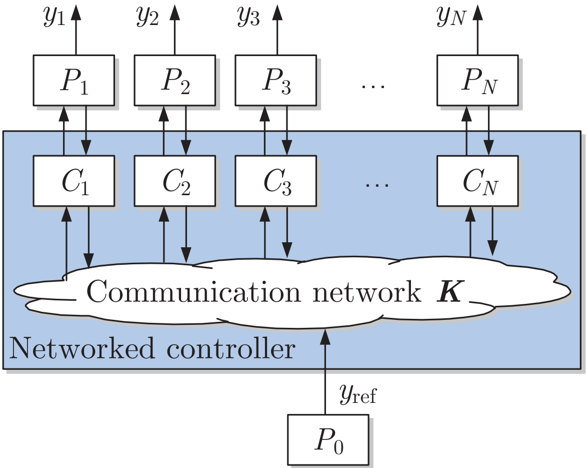
Network Science analyses complex „static“ networks that evolve without fixed organisational structure.

→ Networked control systems:
Use communication structures with small characteristic path length

Design method for the information structure of networked controllers

Communication structure design

Control aim:
Synchronisation with short transient behavour



Communication structure design

Control aim:
Synchronisation with short transient behaviour

Assumptions:

- The leader prescribes a piecewise constant reference trajectory:

$$y_{\text{ref}}(t) = \bar{w}, \quad t \geq 0$$

- The communication is restricted to cycle-free structures.

- The agents may have individual dynamics:

$$P_i : \begin{cases} \dot{\boldsymbol{x}}_i(t) &= \boldsymbol{A}_i \boldsymbol{x}_i(t) + \boldsymbol{b}_i u_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) &= \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{x}_i(t) \end{cases}$$

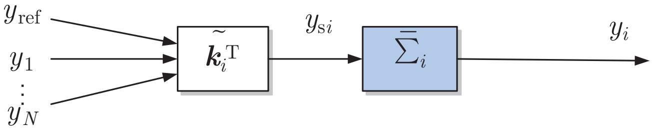
Networked controller:

$$C_i: \quad u_i(t) = -k_i \left(\sum_{j \in \mathcal{P}_i} \tilde{k}_{ij} (y_i(t) - y_j(t)) \right)$$

$$= -k_i \left(y_i(t) - \underbrace{\sum_{j \in \mathcal{P}_i} \tilde{k}_{ij} y_j(t)}_{y_{\mathrm{s}i}(t)} \right), \quad i = 1, 2, ..., N$$

with $\sum_{j \in \mathcal{P}_i} \tilde{k}_{ij} = 1$.

Controlled agent:

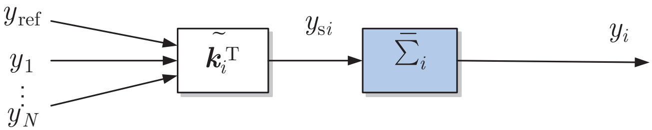


$$\bar{\Sigma}_i : \begin{cases} \dot{\boldsymbol{x}}_i(t) = \bar{\boldsymbol{A}}_i \boldsymbol{x}_i(t) + \bar{\boldsymbol{b}}_i y_{{\rm s}i}(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) = \bar{\boldsymbol{c}}_i^{\rm T} \boldsymbol{x}_i(t) \end{cases}$$

$$y_{{\rm s}i}(t) = \sum_{j \in \mathcal{P}_i} \tilde{k}_{ij} y_j(t)$$

Communication structure design

Synchronisability of the agents:



- **Internal-reference principle:**
  (Lunze, *IEEE TAC* 2012):

  The step response $\bar{h}(t)$ of $\bar{\Sigma}_i$ satisfies the condition

$$\lim_{t \to \infty} \bar{h}(t) = 1.$$

Communication structure design

Synchronisation condition:

The overall system is asympotically synchronised, if

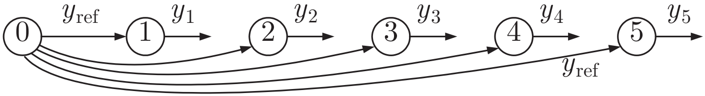
- *all controlled agents* $\bar{\Sigma}_i$, $(i = 1, 2, ..., N)$ *are asymptotically stable,*
- *the communication graph* $\mathcal{G}$ *is cycle-free and includes a spanning tree with the root node 0.*

The freedom in choosing the communication graph should be used to make the transient behaviour as quick as possible.

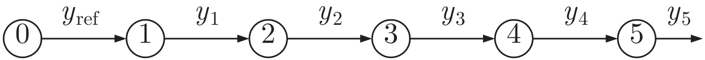
Communication structure design

Examples: Cycle-free communication graphs that include a spanning tree

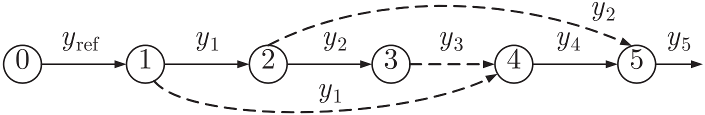
(a)



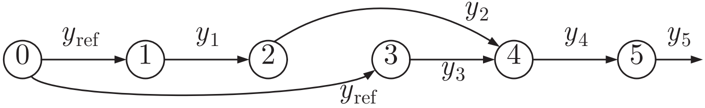
(b)



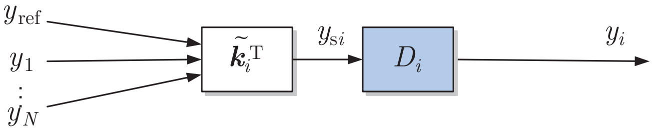
(c)



(d)



Communication structure design



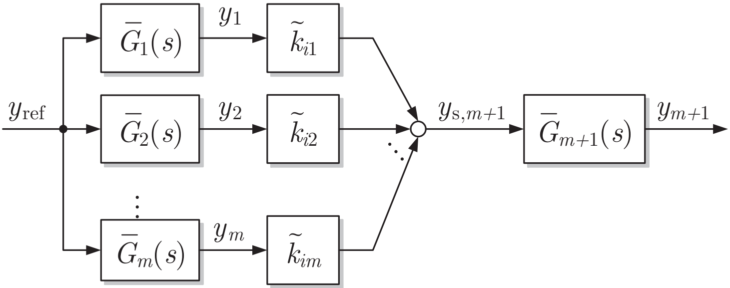
Main idea: *Representation of the agent by a delay:*

$$D_i = \int_0^\infty \left(1 - \bar{h}_i(\tau)\right) \mathrm{d}\tau$$

with $\bar{h}_i(t)$ denoting the step response of $\bar{\Sigma}_i$.

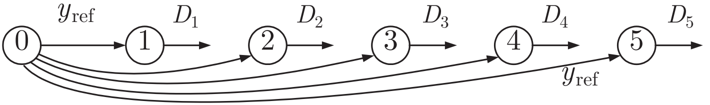
Communication structure design

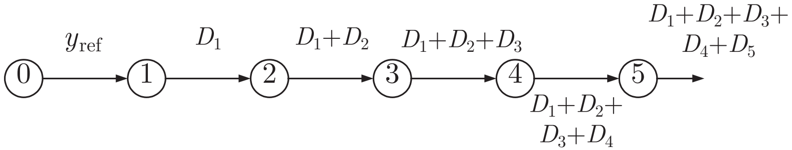
Cumulative delay of a networked system:
(Lunze, *Int. J. Control* 2013)



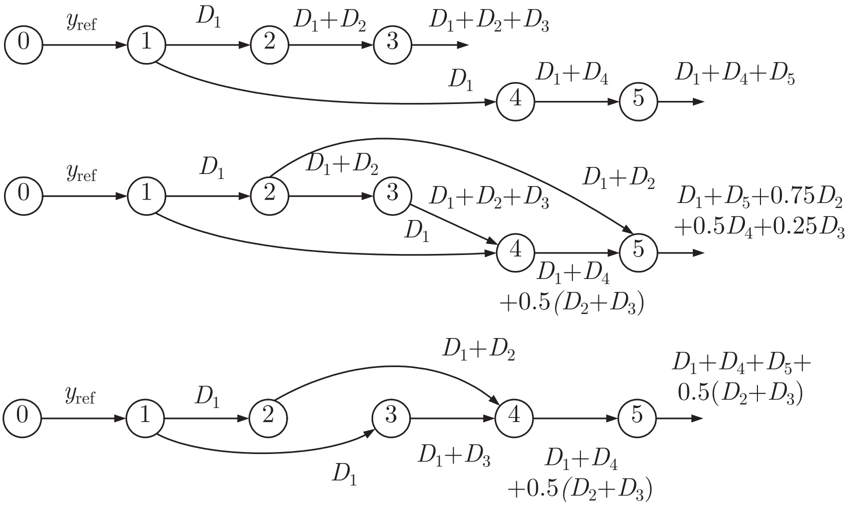
$$D_{i,\mathrm{ref}} = D_{m+1} + \sum_{j=1}^{m} \tilde{k}_{ij} D_{j,\mathrm{ref}}.$$

Communication structure design





Communication structure design



Communication structure design

Formalisation of the design problem:

$$\boldsymbol{D}_{\text{cum}} = \begin{pmatrix} D_{1,\text{ref}} \\ D_{2,\text{ref}} \\ \vdots \\ D_{N,\text{ref}} \end{pmatrix} \quad \text{and} \quad \boldsymbol{D}_{\text{indiv}} = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{pmatrix}$$

$$\boldsymbol{D}_{\text{cum}} = \tilde{\boldsymbol{K}}_{\text{F}} \boldsymbol{D}_{\text{cum}} + \boldsymbol{D}_{\text{indiv}}$$

$$\boldsymbol{D}_{\text{cum}} = \left(\boldsymbol{I} - \tilde{\boldsymbol{K}}_{\text{F}} \right)^{-1} \boldsymbol{D}_{\text{indiv}}$$

Communication structure design

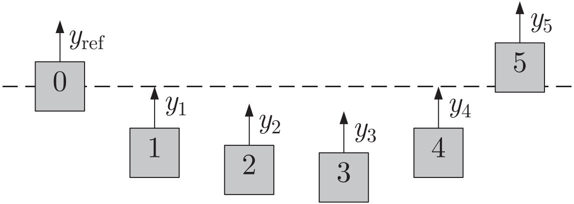
Formalisation of the design problem:

Objective function:

$$J_{\mathrm{OS}}(\tilde{\boldsymbol{K}}_{\mathrm{F}}) = (1,\ 1,\ \ldots 1)\left(\boldsymbol{I} - \tilde{\boldsymbol{K}}_{\mathrm{F}}\right)^{-1} \boldsymbol{D}_{\mathrm{indiv}}.$$

$$\min_{\tilde{\boldsymbol{K}}_{\mathrm{F}} \in \mathcal{K}_{\mathrm{F}}} \quad J_{\mathrm{OS}}$$

Example: Robot formation problem



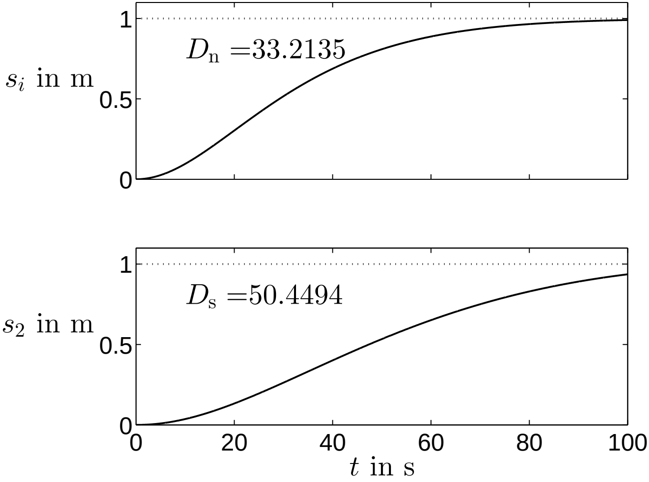
$y_{\text{ref}}(t) = \bar{w}$

Control aims:

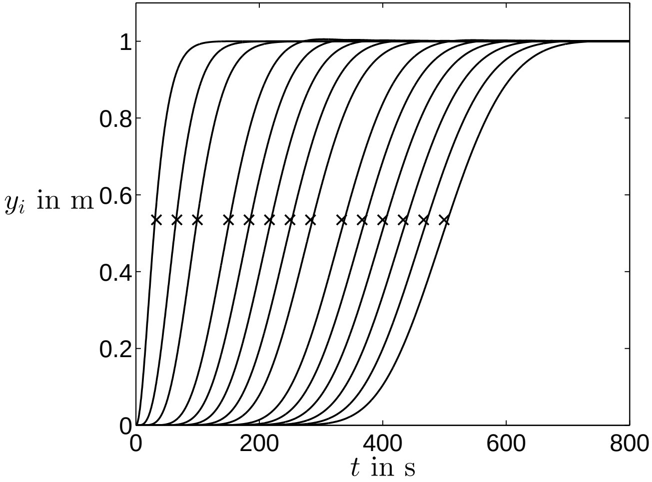
- Asymptotic synchronisation: $\lim_{t\to\infty} |\bar{w} - y_i(t)| = 0$
- Good transient behaviour:

$$J_i = \int_0^\infty (\bar{w} - y_i(t))\, \mathrm{d}t, \quad i = 1, 2, , ..., N.$$

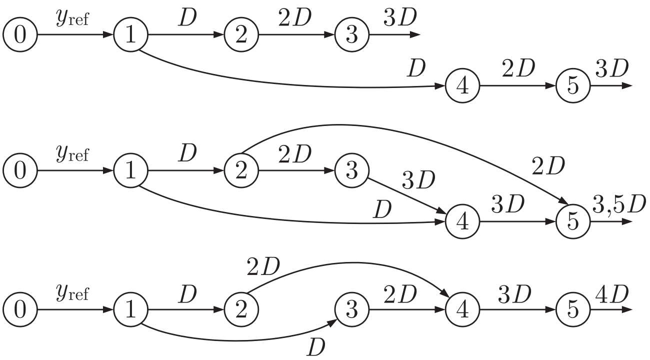
Two kind or robots:

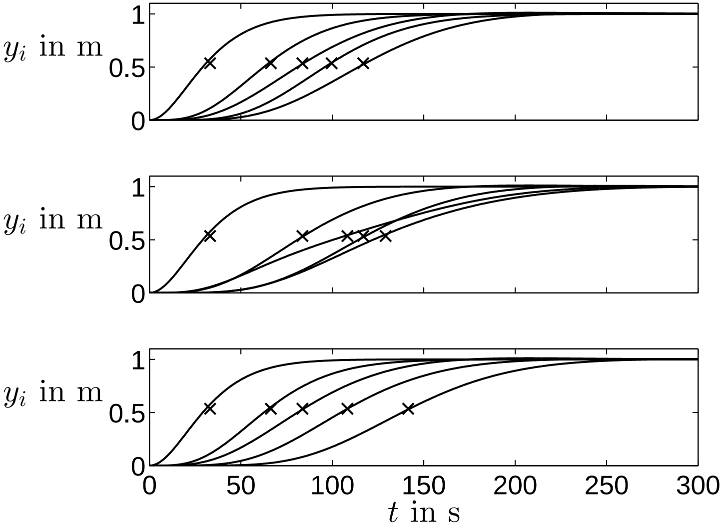


Neighbouring couplings:



Communication structure design
Example: Robot formation problem





1
0.5
0
y_i in m
1
0.5
0
y_i in m
1
0.5
0
y_i in m
0
50
100
150
200
250
300
t in s

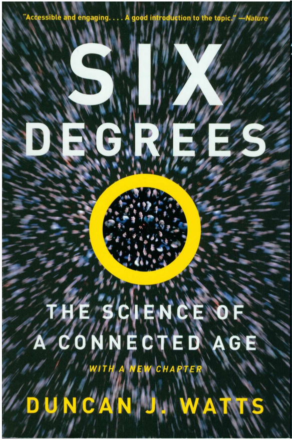
Summary

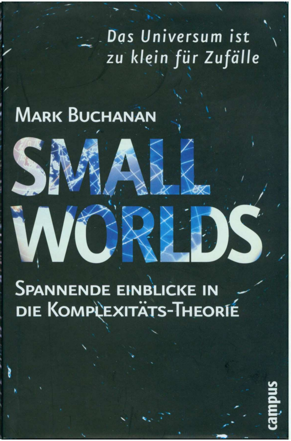
Summary

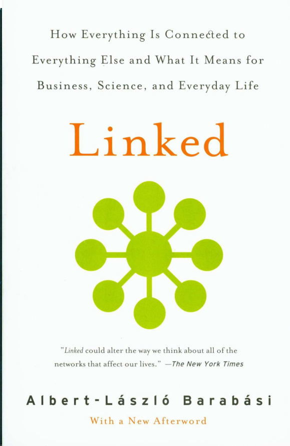
- Large networks are characterised by short characteristic path length.
- For networked systems, the path length is the weighted sum of the delay of the agents.
- Quick transient behaviour of synchronised systems occur if the communication structure leads to short paths from the leader to the followers.

Summary

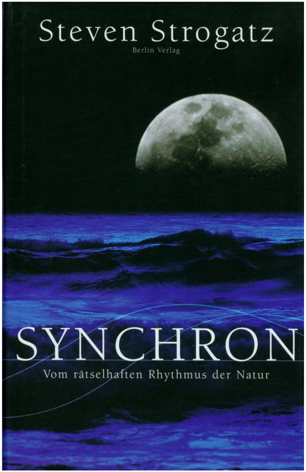
Literature (1):







Literature (2):



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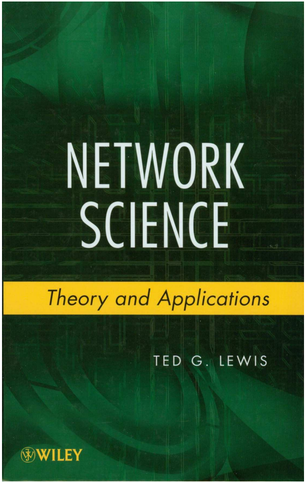
SYNCHRON: Vom rätselhaften Rhythmus der Natur,
Berlin-Verlag 2004.

Summary

Literature (3):



Lewis, T. G.:
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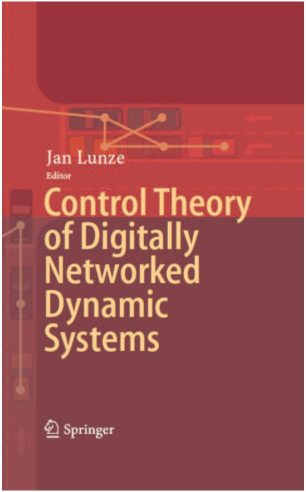
Summary

Literature (4):



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