Research Supported by NSF & ARO

Networked Control Systems

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Center for Control, Dynamical Systems and Computation CCDC

Networked Control Systems

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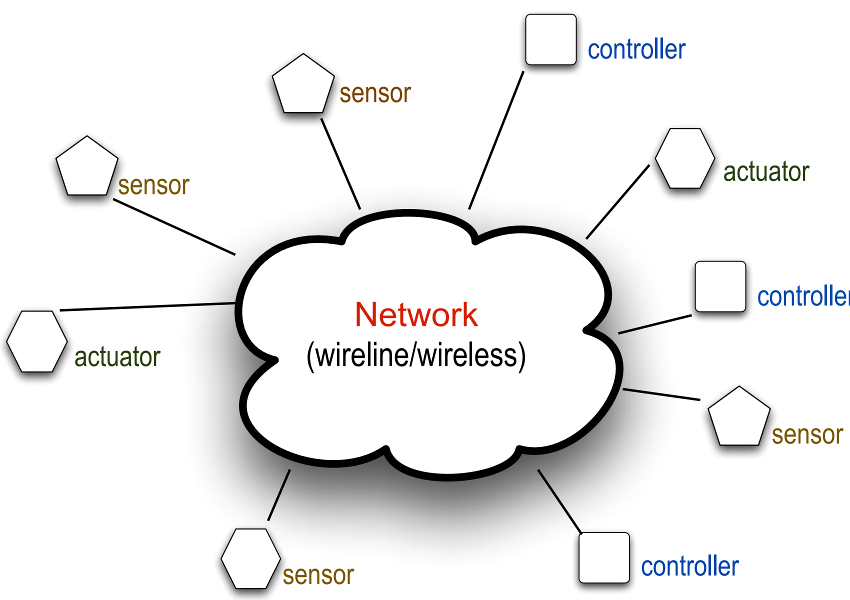

Application Areas

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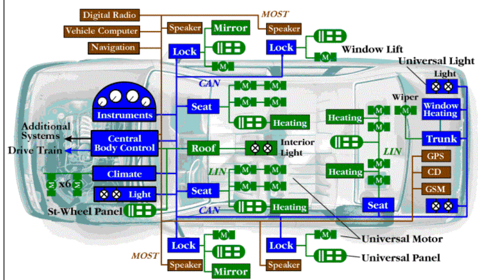
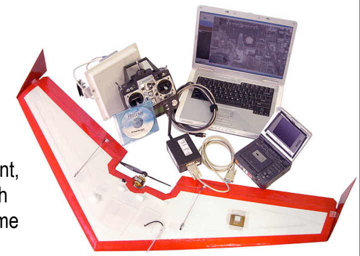
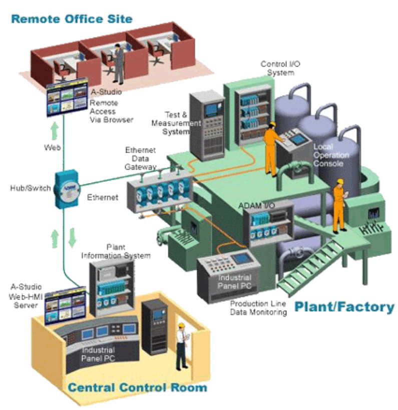


Efficiency and safety in cars depend on a network of hundreds of ECUs (power train, ABS, stability control, speed control, transmission, …)

Robotic agents free humans from unpleasant, dangerous, and/or repetitive tasks in which human performance would degrade over time due to fatigue





Process control or power plant facilities often have between several thousand of coupled control loops

Buildings consume 72% of electricity, 40% of all energy, and produce close to 50% of U.S. carbon emissions

Challenges

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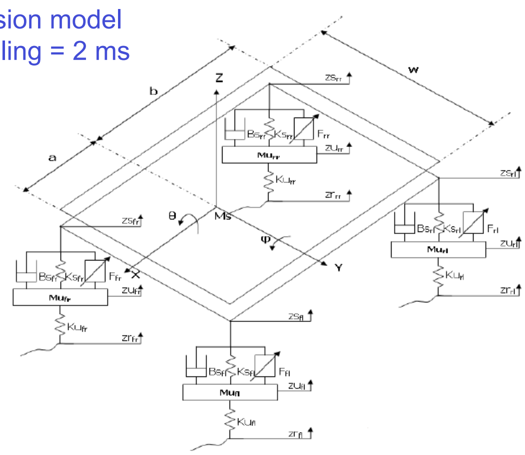
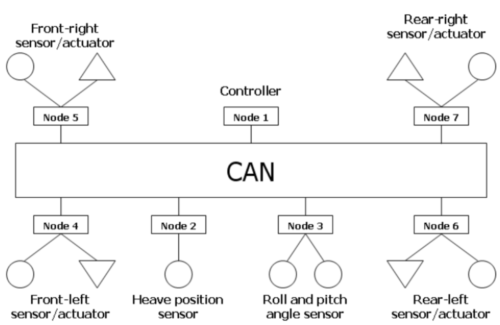
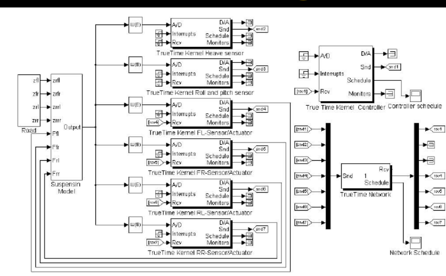
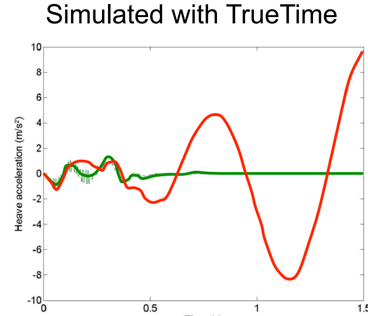
Active suspension model
constant sampling = 2 ms







Simulated with TrueTime



node	1 2 3 4 5 6 7
priorities 1	1 2 3 4 5 6 7
priorities 2	7 1 2 3 4 5 6

network access priorities

Ben Gaid, Cela, Kocik

Digital Control Systems

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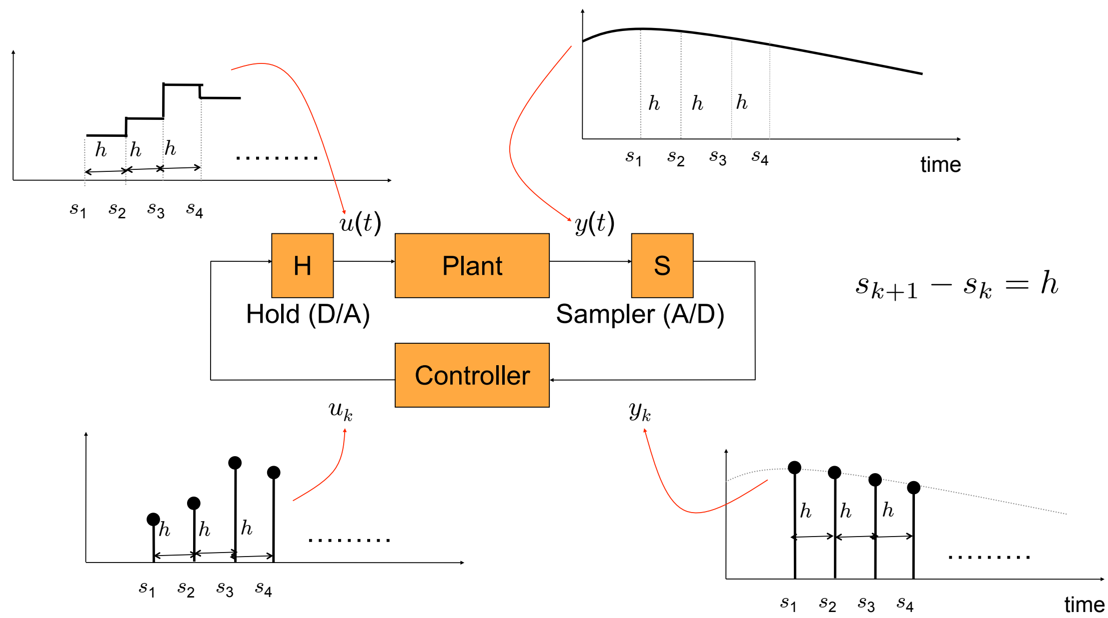
Digital control systems usually exhibit uniform sampling intervals and delays



$$s_{k+1} - s_k = h$$

Non-uniform Sampling/Delays

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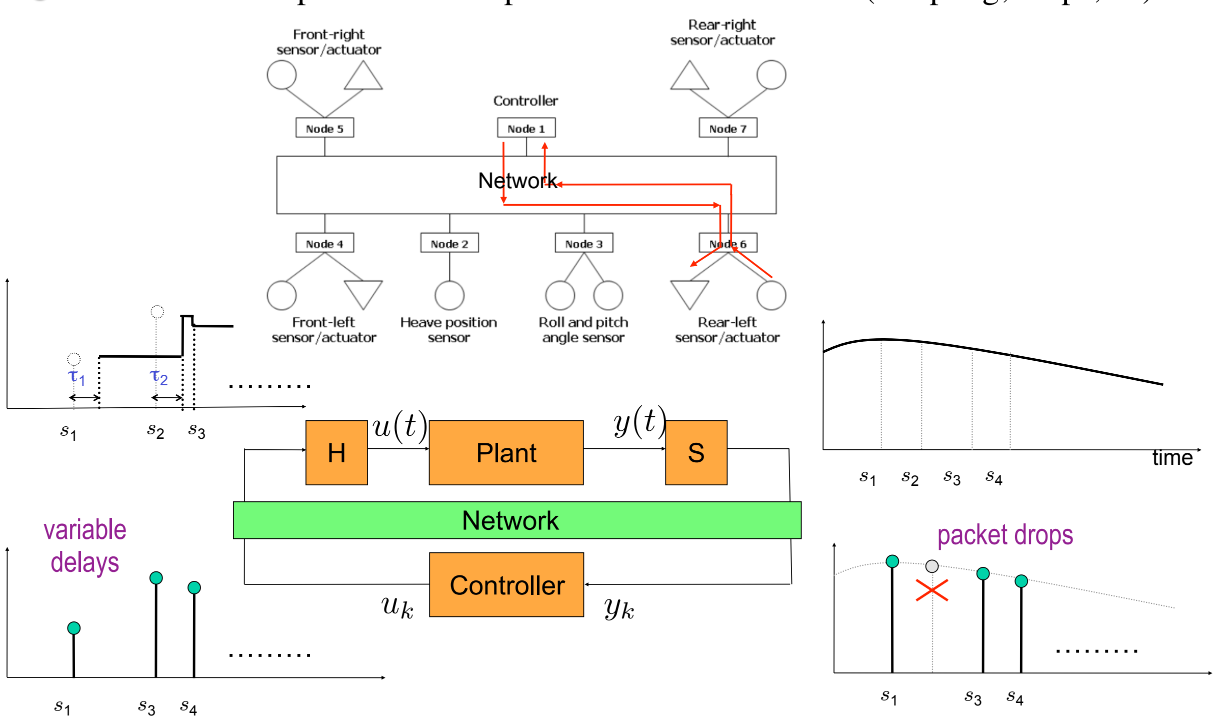
- Uniform sampling cannot be guaranteed (packet drops, clock synchronization, …)
- Different samples may experience different delays
- Difficult to decouple continuous plant from discrete events (sampling, drops, …)



Course Overview

Lecture #1: Modeling Framework – Hybrid Dynamical Systems
(Deterministic, Stochastic, Impulsive)

Lecture #2: Analysis of Stochastic Hybrid Systems
(Generator, Lyapunov-based Methods)

(extra material): NCS Protocol Design
(Medium Access, Transport, Routing)

Lecture #1
Modeling Framework:
Hybrid Dynamical Systems

Lecture #1 Outline

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- Deterministic Impulsive Systems (DISs)
- Deterministic Hybrid Systems (DHSs)
- Stochastic Hybrid Systems (SHSs)
- Simulation of SHSs
- SHSs Driven by Renewal Processes

Main references:
Davis, "Markov Models and Optimization" Chapman & Hall,1993
Cassandras, Lygeros, "SHSs" CRC Press 2007
Hespanha, "A Model for SHSs with Application ..." Nonlinear Analysis 2005.

Deterministic Impulsive Systems

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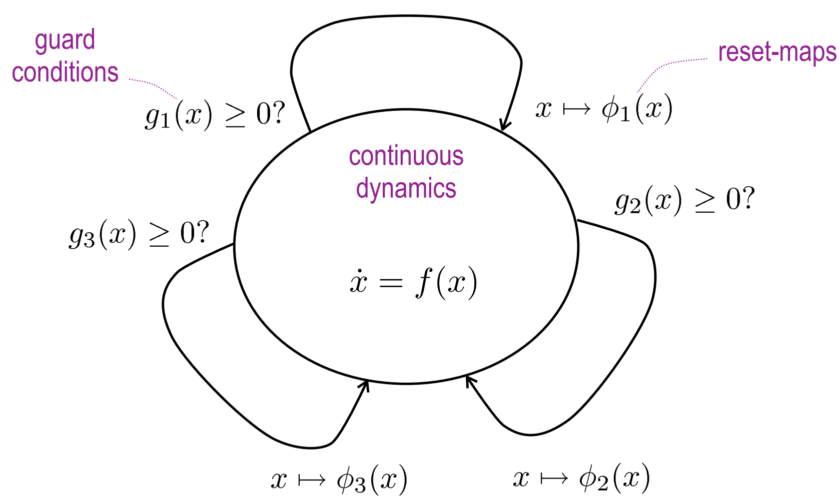
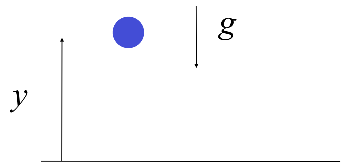


$x(t) \in \mathbb{R}^n \equiv$ continuous state

Example #1: Bouncing Ball



Free fall $\equiv \quad \ddot{y} = -g$

Collision $\equiv \quad y^{+}(t) = y^{-}(t) = 0$

$$\dot{y}^{+}(t) = -c\dot{y}^{-}(t)$$

$c \in [0,1] \equiv$ energy "reflected" at impact

Notation: given $x : [0,\infty) \to \mathbb{R}^n \equiv$ piecewise continuous signal

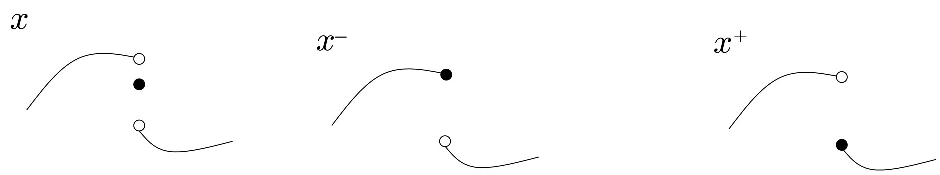
$$x^{-} : (0,\infty) \to \mathbb{R}^n \qquad x^{-}(t) := \lim_{\tau \uparrow t} x(t), \quad \forall t > 0$$

$$x^{+} : [0,\infty) \to \mathbb{R}^n \qquad x^{+}(t) := \lim_{\tau \downarrow t} x(t), \quad \forall t \geq 0$$

at points t where x is continuous $x(t) = x^{-}(t) = x^{+}(t)$

By convention we will generally assume right continuity, i.e.,

$$x(t) = x^{+}(t) \quad \forall t \geq 0$$



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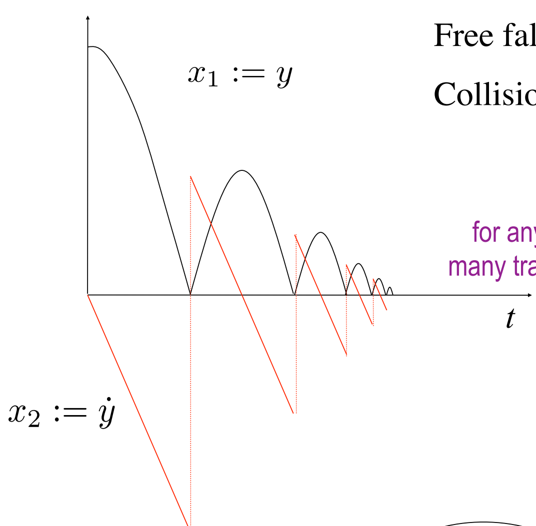

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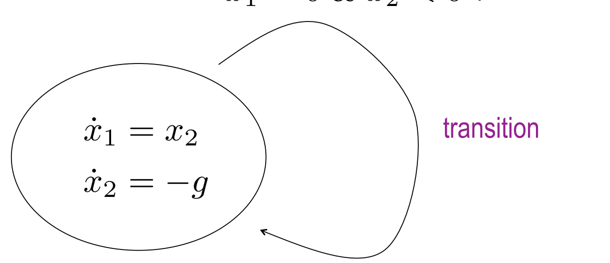
Collision $\equiv \quad y^{+}(t) = y^{-}(t) = 0$

$$\dot{y}^{+}(t) = -c\dot{y}^{-}(t)$$

for any c < 1,there are infinitely many transitions in finite time (Zeno phenomena)

guard or jump condition

$x_1 = 0 \;\&\; x_2 < 0$?

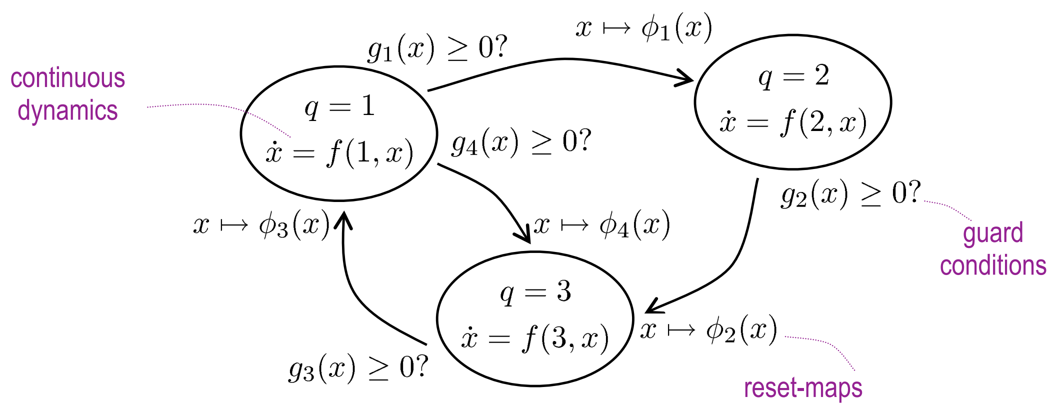


$x_2 \mapsto -cx_2^{-}$

state reset

Impulsive System
(all discreteness in the form of instantaneous changes in the state)

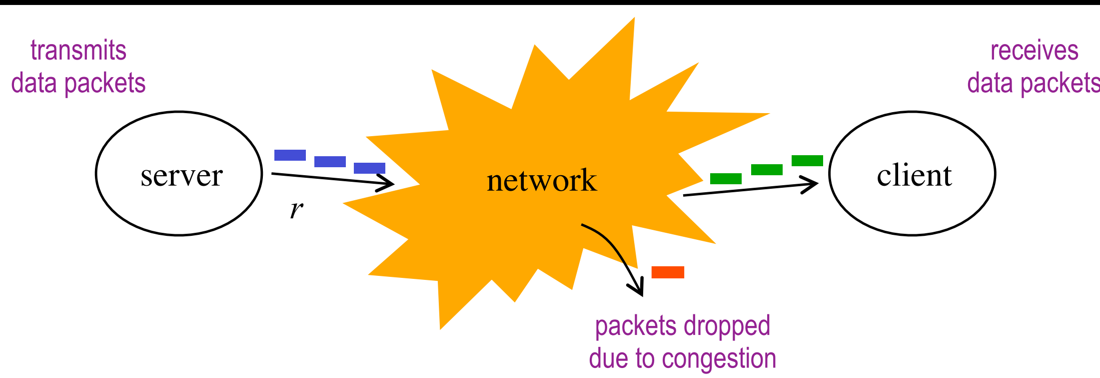
Deterministic Hybrid Systems



$q(t) \in \mathcal{Q}=\{1,2,\dots\}$ $\equiv$ discrete state
$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

right-continuous by convention

Example #2: TCP Congestion Control



congestion control $\equiv$ selection of the rate r at which the server transmits packets

feedback mechanism $\equiv$ packets are dropped by the network to indicate congestion

TCP (Reno) congestion control: packet sending rate given by

$$r(t) = \frac{w(t)}{RTT(t)}$$

congestion window (internal state of controller)

round-trip-time (from server to client and back)

- initially w is set to 1
- until first packet is dropped, w increases exponentially fast (slow-start)
- after first packet is dropped, w increases linearly (congestion-avoidance)
- each time a drop occurs, w is divided by 2 (multiplicative decrease)

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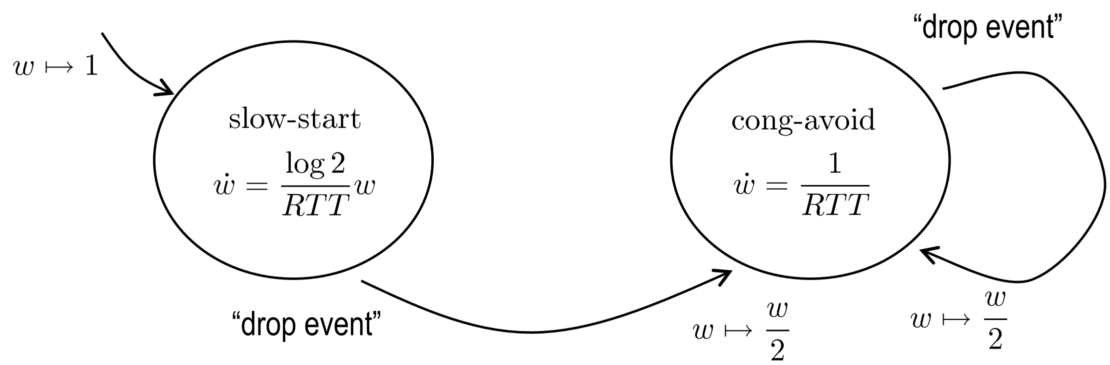

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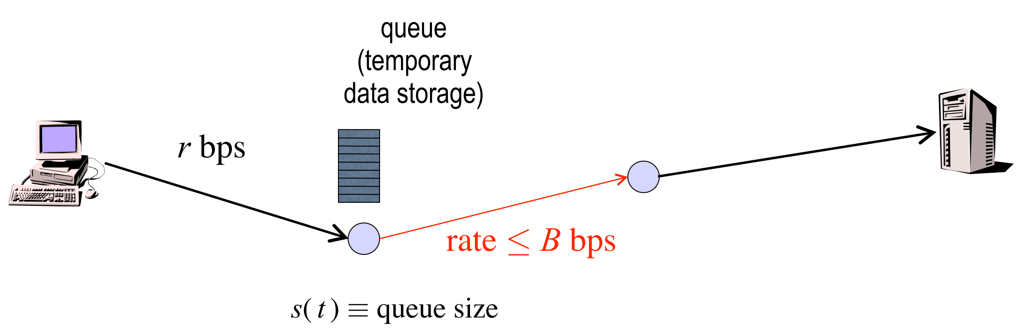
Drops by Queue Overflow



When r exceeds B the queue fills and data is lost (drops)

$$\dot{s} = \begin{cases} r - B & 0 \leq s \leq s_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$s = s_{\max}, \quad r > B \qquad \Rightarrow \text{ drop event}$$

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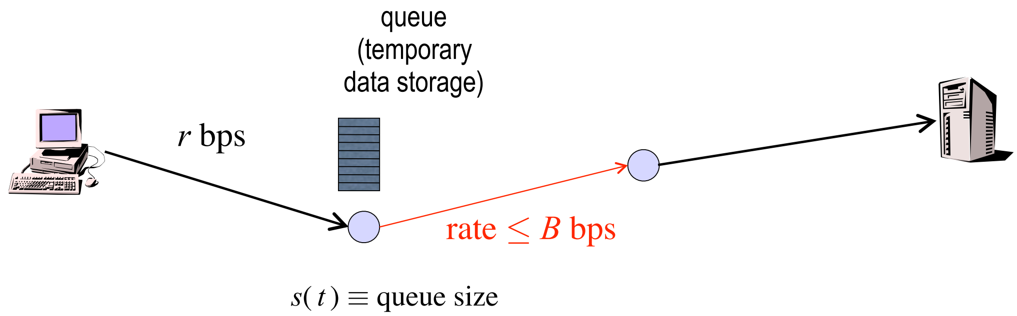
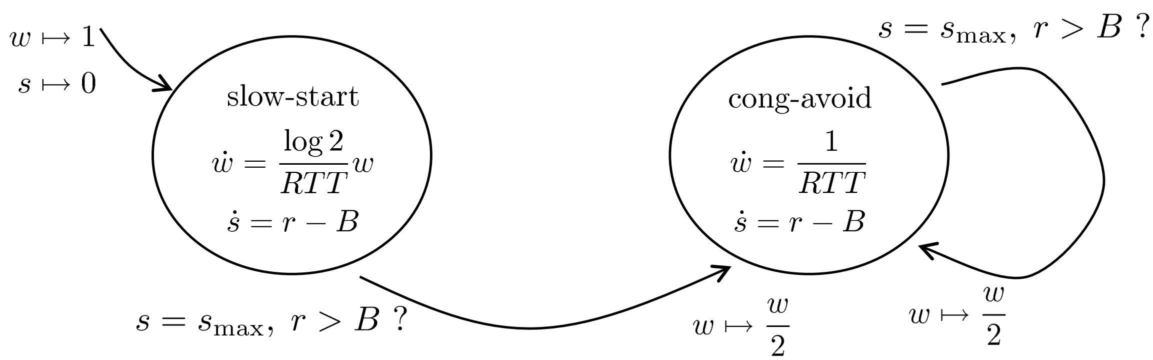
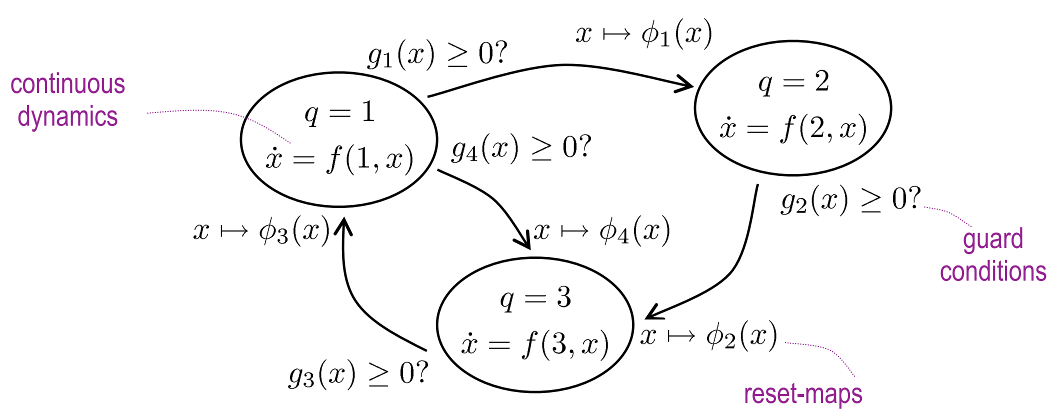




So far…



$$\left.\begin{array}{lll} q(t) \in \mathcal{Q}=\{1,2,\ldots\} & \equiv \text{discrete state} \\ x(t) \in \mathbb{R}^n & \equiv \text{continuous state} \end{array}\right\} \text{right-continuous by convention}$$

Stochastic Hybrid Systems

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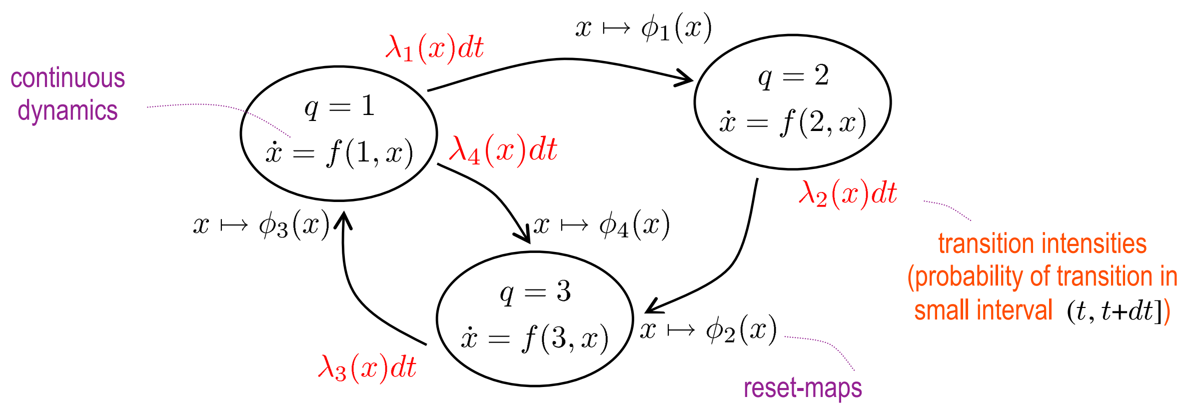

$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state

$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

$\lambda_\ell(x)dt$ $\equiv$ probability of transition in an "elementary" interval $(t, t+dt]$

$$\Downarrow$$

$\lambda_\ell(x)$ $\equiv$ instantaneous rate of transitions per unit of time

Stochastic Hybrid Systems

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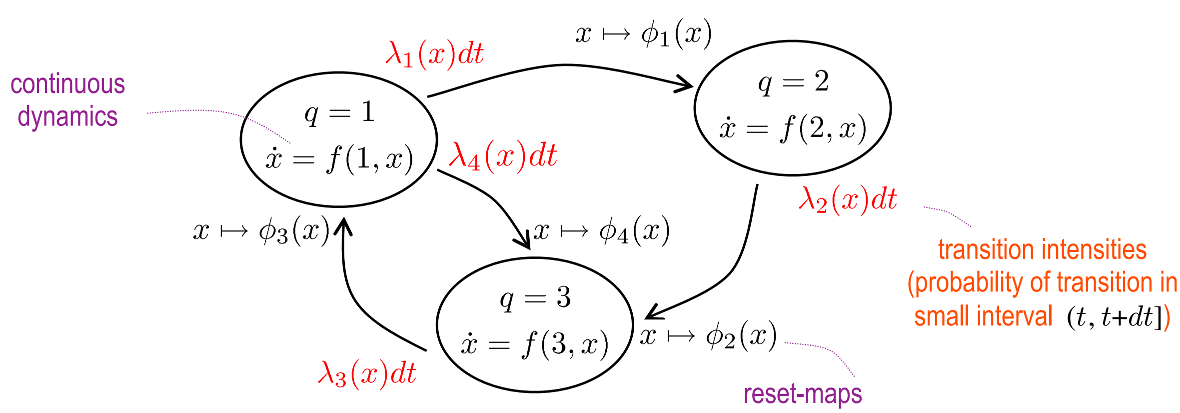
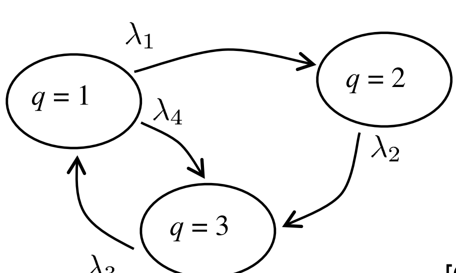


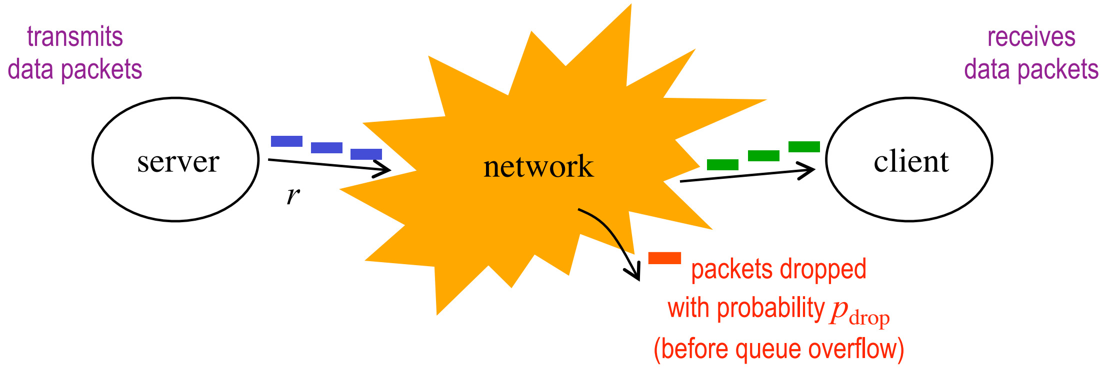
Special case: When all λ_ℓ are constant, transitions are controlled by a continuous-time Markov process



specifies q
(independently of x)

closely related to the so called *Markovian Jump Systems* [Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]

Example #2.1: TCP Congestion Control



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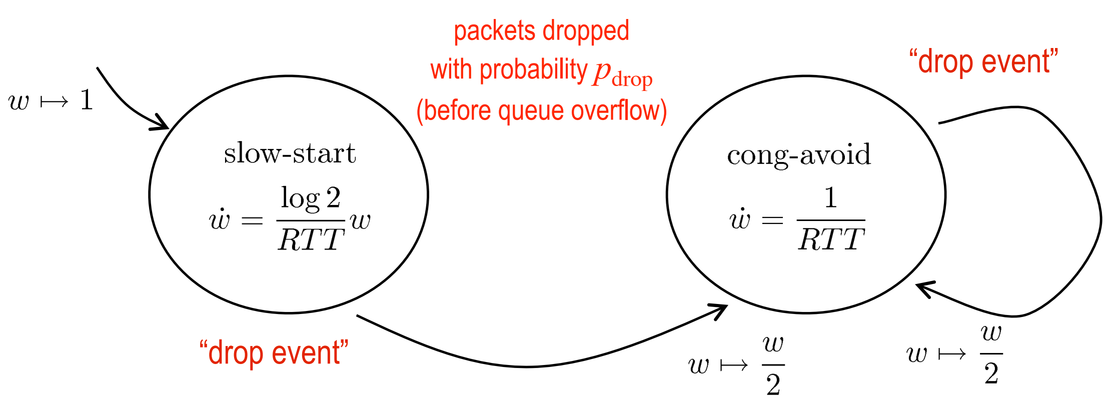

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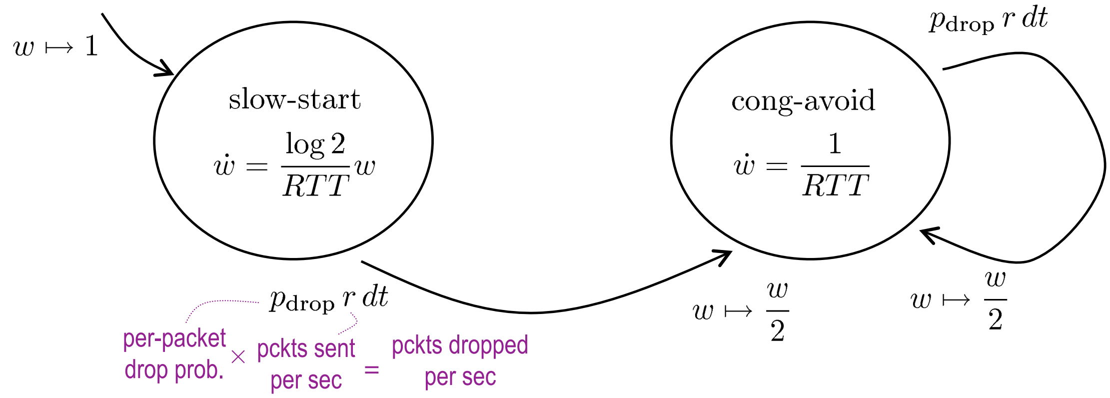

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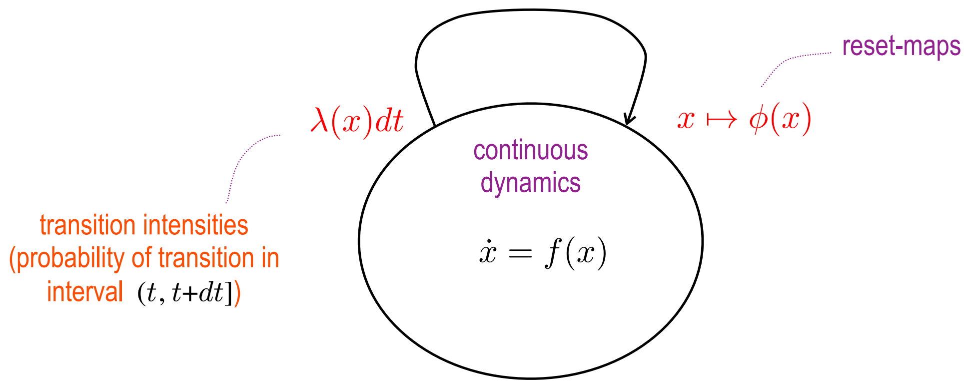
Stochastic Impulsive Systems



Numerical Simulation of SISs

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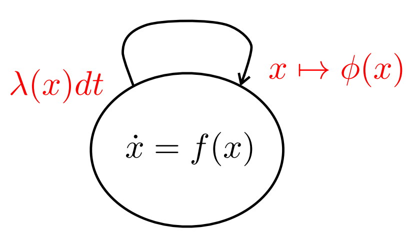


here we take x_0 as a given parameter

1. Initialize state:

$$x(t_0) = x_0 \quad k = 0$$

2. Draw a unit-mean exponential random variable

$$E \sim \exp(1)$$

3. Solve ODE

$$\dot{x} = f(x) \quad x(t_k) = x_k \quad t \geq t_k$$

until time t_{k+1} for which

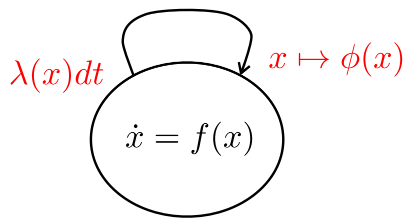
$$\int_{t_k}^{t_{k+1}} \lambda(x(t))dt \geq E$$

4. Apply the corresponding reset map

$$x(t_{k+1}) = x_{k+1} := \phi(x^-(t_{k+1}))$$

set $k = k + 1$ and go to 2.

Numerical Simulation of SISs



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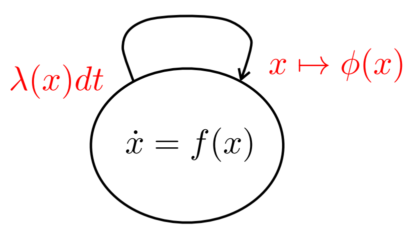
Why does this algorithm lead to

$\lambda(x)$ ≡ instantaneous rate of transitions per unit of time ?

Numerical Simulation of SISs



Solve ODE
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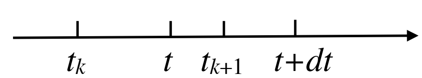

$$\mathrm{P}\Big(\text{jump in } (t, t+dt] \;\Big|\; t_k, x(t_k), \text{no jump in } [t_k, t]\Big)$$
$$= \mathrm{P}\left(\int_{t_k}^{t} \lambda < E \leq \int_{t_k}^{t+dt} \lambda \;\Big|\; t_k, x(t_k), \int_{t_k}^{t} \lambda < E\right)$$
$$= \frac{\mathrm{P}\left(\int_{t_k}^{t} \lambda < E \leq \int_{t_k}^{t+dt} \lambda \mid t_k, x(t_k)\right)}{\mathrm{P}\left(\int_{t_k}^{t} \lambda < E \mid t_k, x(t_k)\right)}$$
$$= \frac{e^{-\int_{t_k}^{t}\lambda} - e^{-\int_{t_k}^{t+dt}\lambda}}{e^{-\int_{t_k}^{t}\lambda}} = 1 - e^{-\int_{t}^{t+dt}\lambda} \xrightarrow{dt \to 0} \lambda(x(t))dt$$

conditional probability

exponential distribution

Numerical Simulation of SISs

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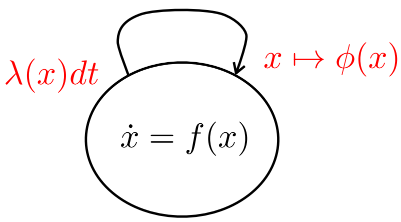

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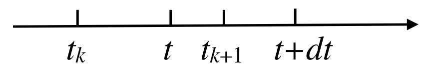

$$\mathrm{P}\Big(\text{jump in } (t, t+dt] \mid t_k, x(t_k), \text{no jump in } [t_k, t]\Big) \xrightarrow{dt \to 0} \lambda(x(t))dt$$

$$\mathrm{P}\Big(\text{multiple jumps in } (t, t+dt] \mid t_k, x(t_k), \text{no jump in } [t_k, t]\Big) = \cdots = O(dt^2)$$

Numerical Simulation of SISs

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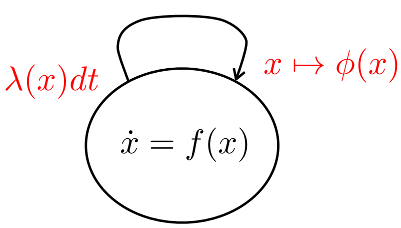

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. Apply the corresponding reset map

$$x(t_{k+1}) = x_{k+1} := \phi(x^-(t_{k+1}))$$

set $k = k + 1$ and go to 2.

This algorithm is “exact” modulo:
- errors in extracting realizations of exponential random variables
- numerical errors in solving ODE
- numerical errors in “zero-crossing” detection

overall very accurate...

Stochastic Impulsive Systems

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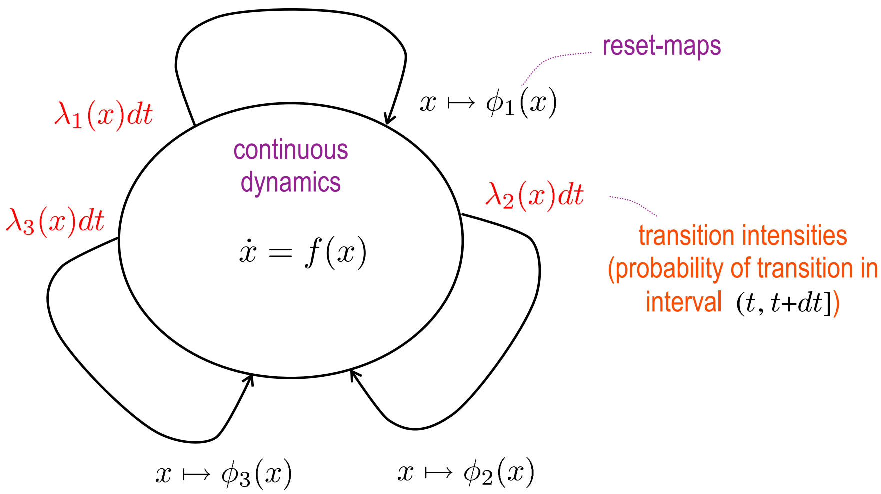

Numerical Simulation of SISs

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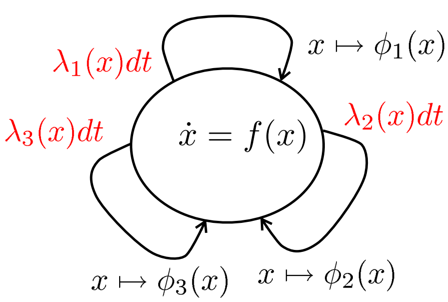

1. Initialize state:

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2. Draw one independent exponential random variable (unit mean) per transition

$$E_1, E_2, E_3 \sim \exp(1)$$

3. Solve ODE

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until time t_{k+1} for which

$$\int_{t_k}^{t_{k+1}} \lambda_\ell(x(t))dt \geq E_\ell$$

for some transition ℓ^*.

4. Apply the corresponding reset map ℓ^*

$$x(t_{k+1}) = x_{k+1} := \phi_{\ell^*}(x^-(t_{k+1}))$$

set $k = k + 1$ and go to 2.

Numerical Simulation of SISs

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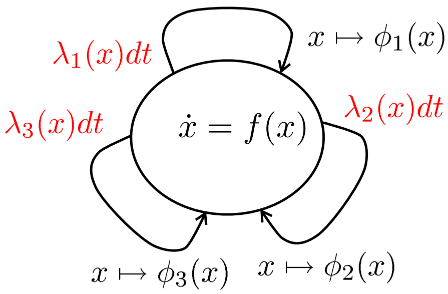

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$$E_1, E_2, E_3 \sim \exp(1)$$
3. Solve ODE
$$\begin{cases} \dot{x} = f(x) & x(t_k) = x_k \\ \dot{m}_1 = \lambda_1(x) & m_1(t_k) = 0 \\ \dot{m}_2 = \lambda_2(x) & m_2(t_k) = 0 \\ \vdots & \vdots \end{cases} \quad t \geq t_k$$
until time t_{k+1} for which
$$m_\ell(t_{k+1}) \geq E_\ell$$
for some transition ℓ^*.
4. Apply the corresponding reset map ℓ^*
$$x(t_{k+1}) = x_{k+1} := \phi_\ell(x^-(t_{k+1}))$$
set $k = k + 1$ and go to 2.

Numerical Simulation of SISs

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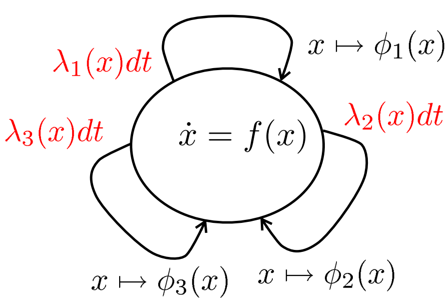

Under appropriate (mild) assumptions this procedure results in a (strong) Markov Process $x(t)$

However…

1. Initialize state:
$$x(t_0) = x_0 \quad k = 0$$
2. Draw one independent exponential random variable (unit mean) per transition
$$E_1, E_2, E_3 \sim \exp(1)$$
3. Solve ODE
$$\begin{cases} \dot{x} = f(x) & x(t_k) = x_k \\ \dot{m}_1 = \lambda_1(x) & m_1(t_k) = 0 \\ \dot{m}_2 = \lambda_2(x) & m_2(t_k) = 0 \\ \vdots & \vdots \end{cases} \quad t \geq t_k$$
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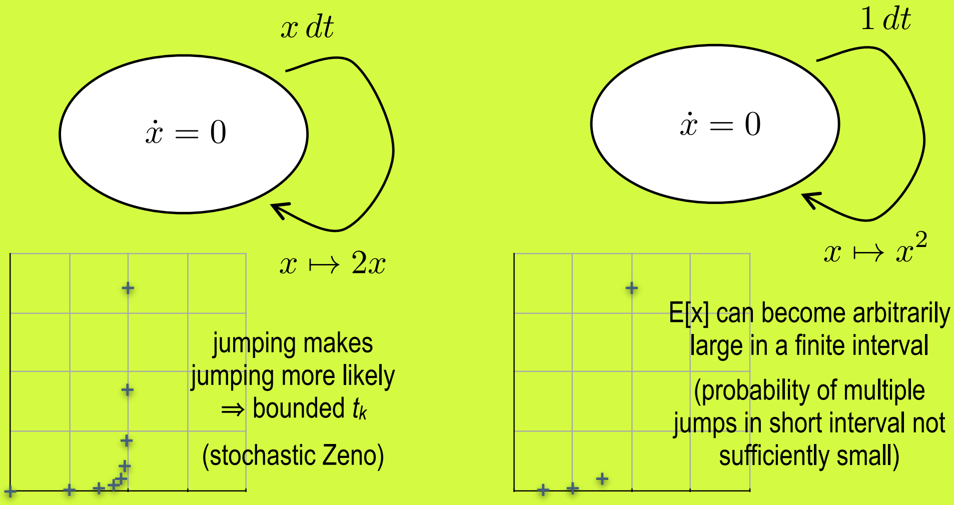
set $k = k + 1$ and go to 2.

Numerical Simulation of SISs

λ_1

$\lambda_3(x)$

Attention:
These systems may have issues with existence of solution due to jumps!
E.g.

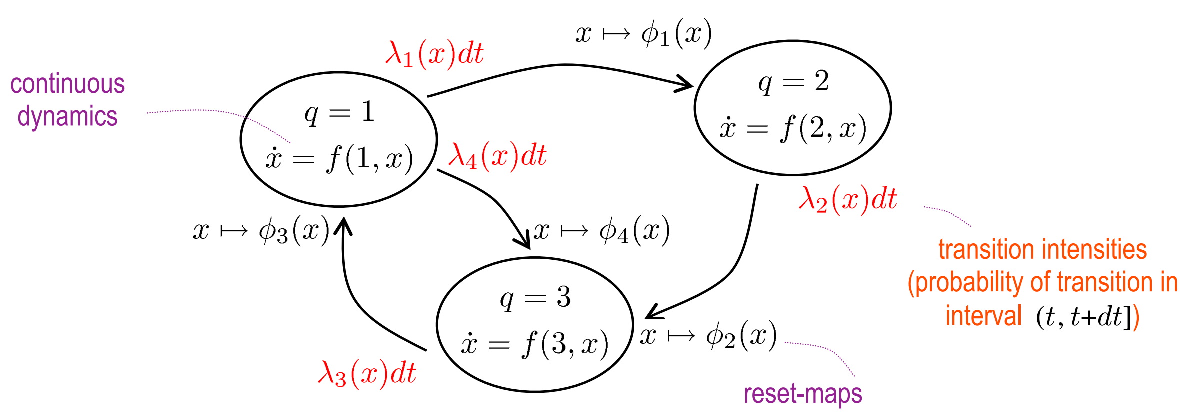


In either case, "bad things can happen" with nonzero probability.

m

and go to 2.

back to Stochastic Hybrid Systems …

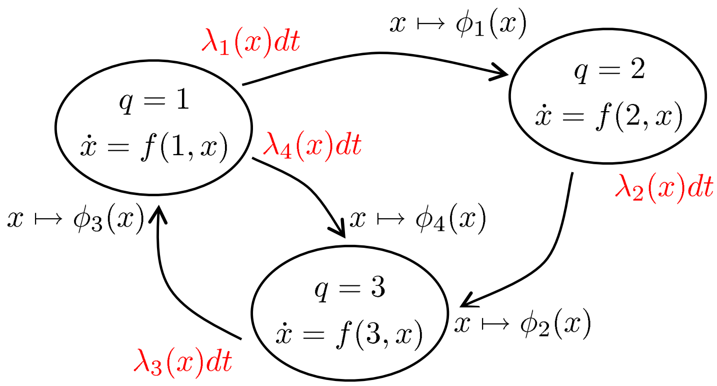


$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ ≡ discrete state
$x(t) \in \mathbb{R}^n$ ≡ continuous state

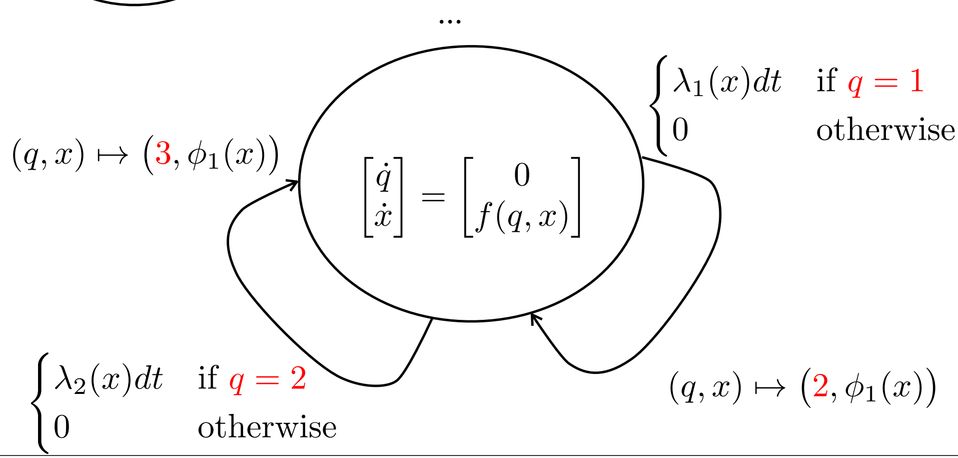
For simulation purposes, we can view the SHS as a SIS with an enlarged state

$$z := \begin{bmatrix} q \\ x \end{bmatrix} \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} \dot{q} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 \\ f(q,x) \end{bmatrix} =: F(z)$$

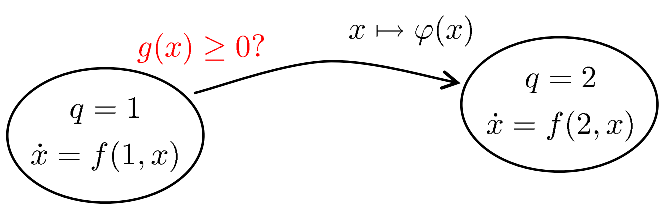
back to Stochastic Hybrid Systems …



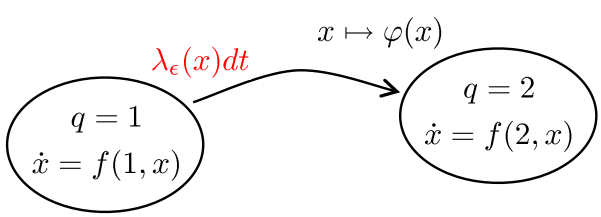
Same algorithms can be used to simulate the equivalent SIS

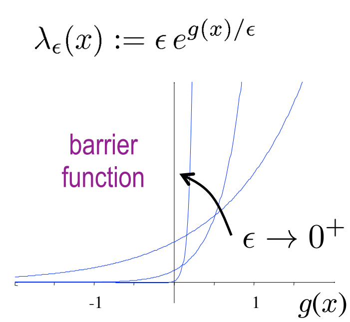


Generalizations



1. *Deterministic guards* can also be emulated by taking limits of SHSs

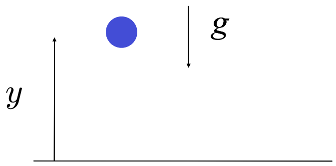


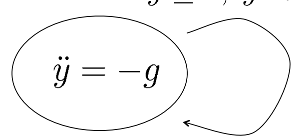


The solution for the deterministic guard is obtained as $\epsilon \to 0^+$

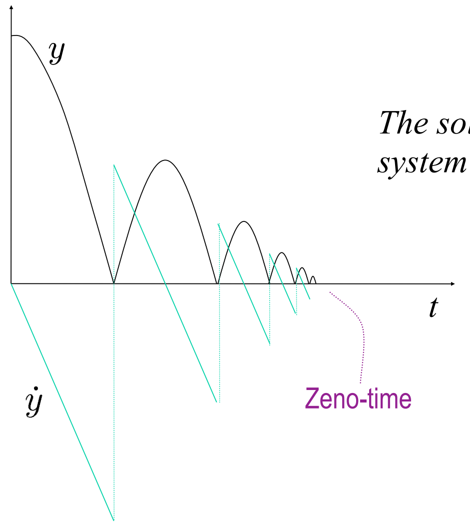
This provides a mechanism to regularize systems with chattering and/or Zeno phenomena…

Example #1: Bouncing-ball





$c \in (0,1) \equiv$ energy absorbed at impact



The solution of this deterministic hybrid system is only defined up to the Zeno-time

Stochastic Bouncing-Ball

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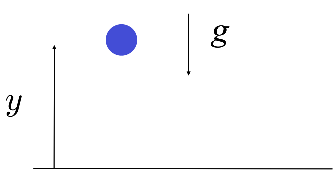
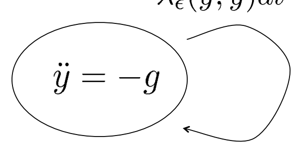
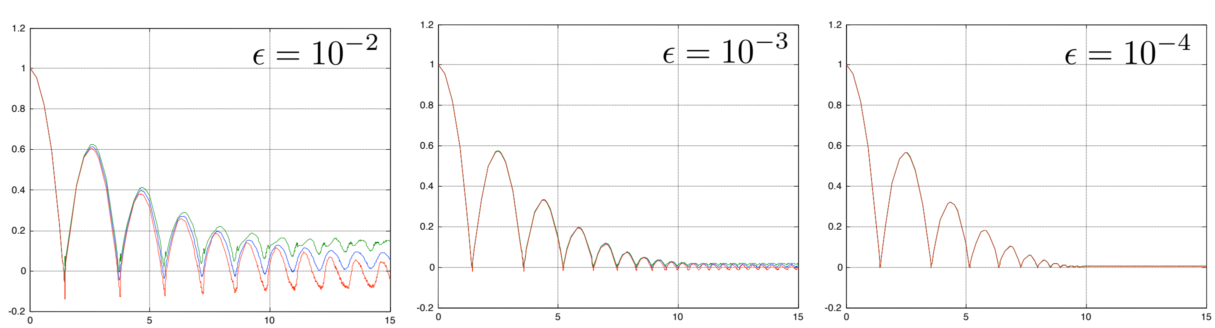




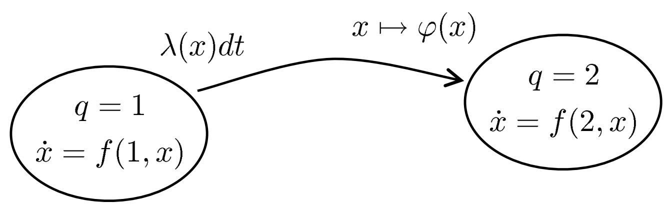
$$\lambda_\epsilon(y, \dot{y}) := \begin{cases} \epsilon e^{-y/\epsilon} & \dot{y} < 0 \\ 0 & \dot{y} > 0 \end{cases}$$

$c \in (0,1) \equiv$ energy absorbed at impact

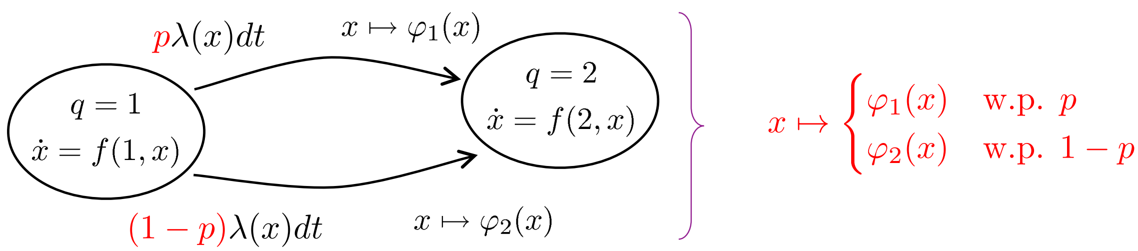


mean (blue) 95% confidence intervals (red and green)

Generalizations

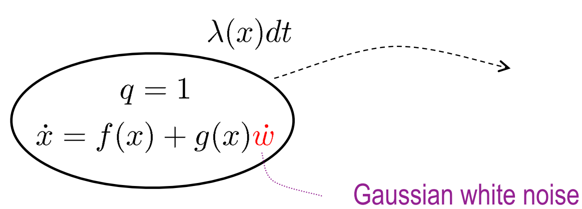


2. *Stochastic resets* can be obtained by considering multiple intensities/reset-maps

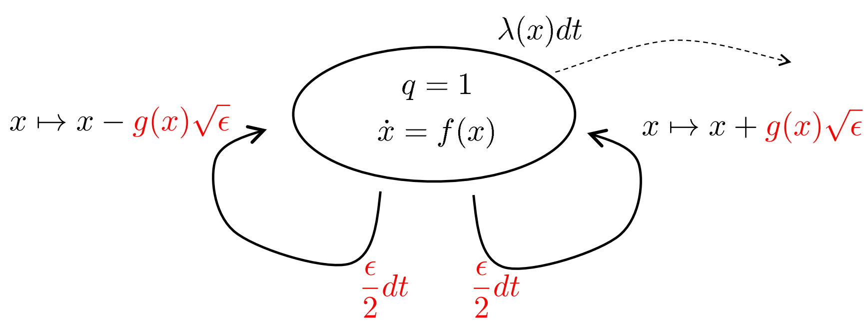


One can further generalize this to resets governed by a continuous distribution
$x \sim \mu(q, x, dx)$

Generalizations

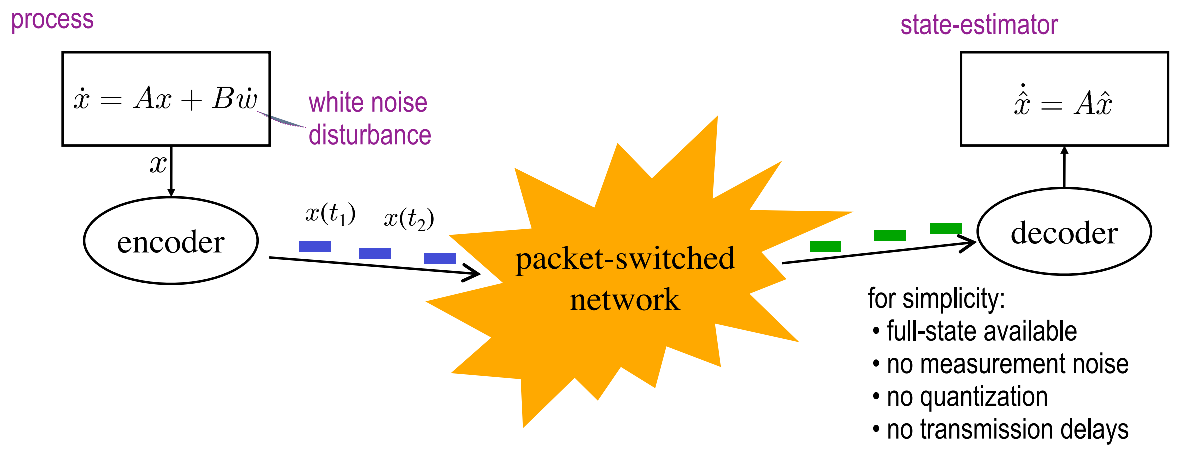


3. *Stochastic differential equations (SDE) for the continuous state* can be emulated by taking limits of SHSs



The solution to the SDE is obtained as $\epsilon \to 0^+$

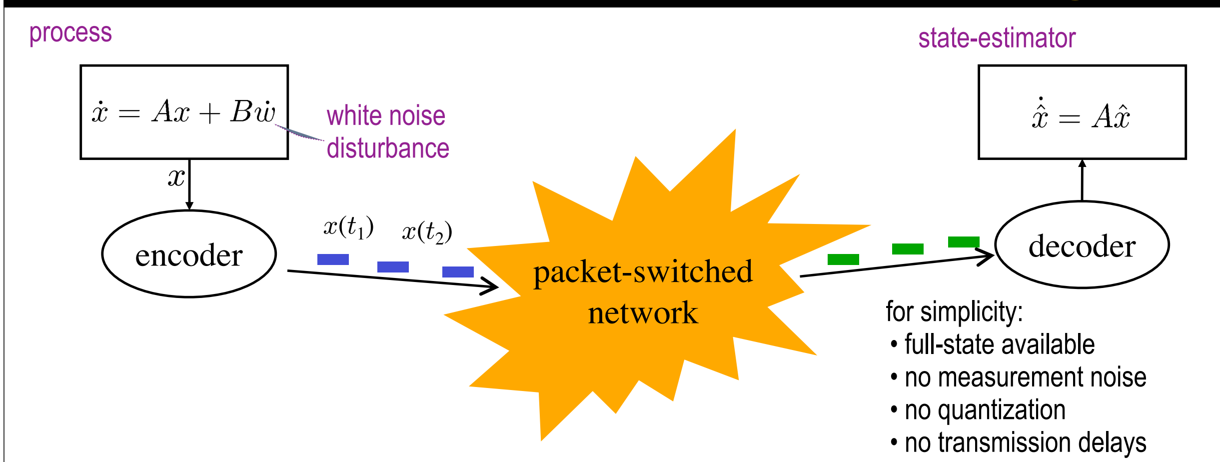
Example #3: Estimation through network



encoder logic $\equiv$ determines *when* to send measurements to the network

decoder logic $\equiv$ determines *how* to incorporate received measurements

Stochastic communication logic



encoder logic $\equiv$ determines *when* to send measurements to the network

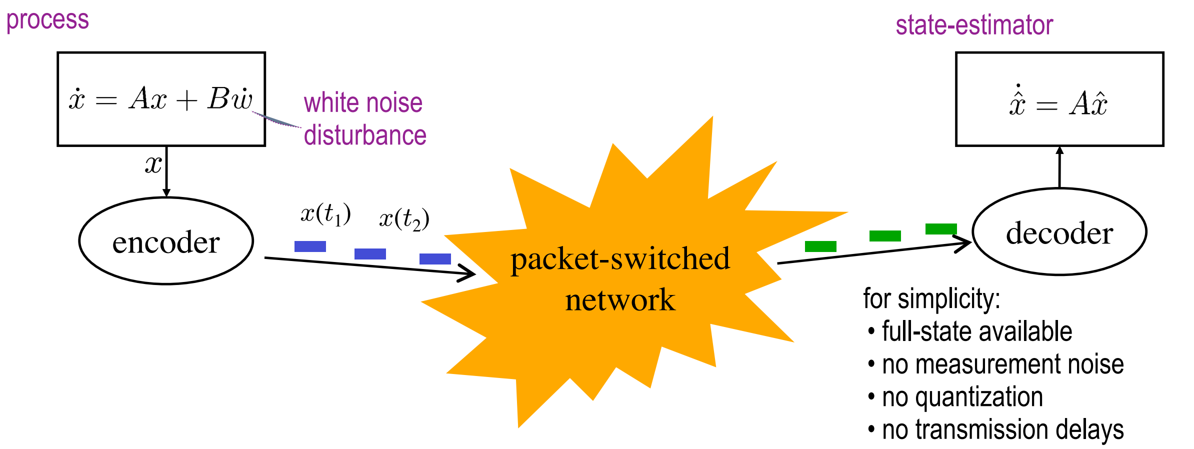
1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from x

decoder logic $\equiv$ determines *how* to incorporate received measurements

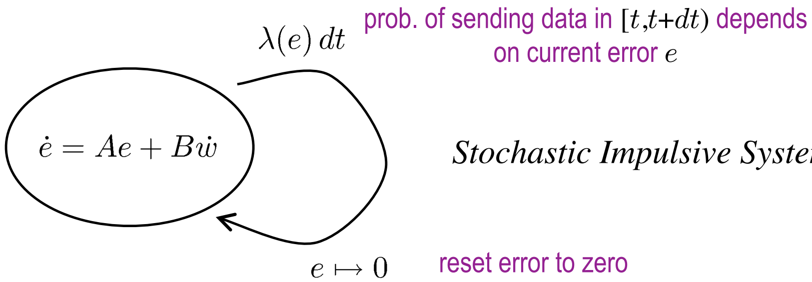
4. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

[similar ideas pursued by Astrom , Tilbury, Hristu, Kumar, Basar]

Error Dynamics



Error dynamics: $e := x - \hat{x}$

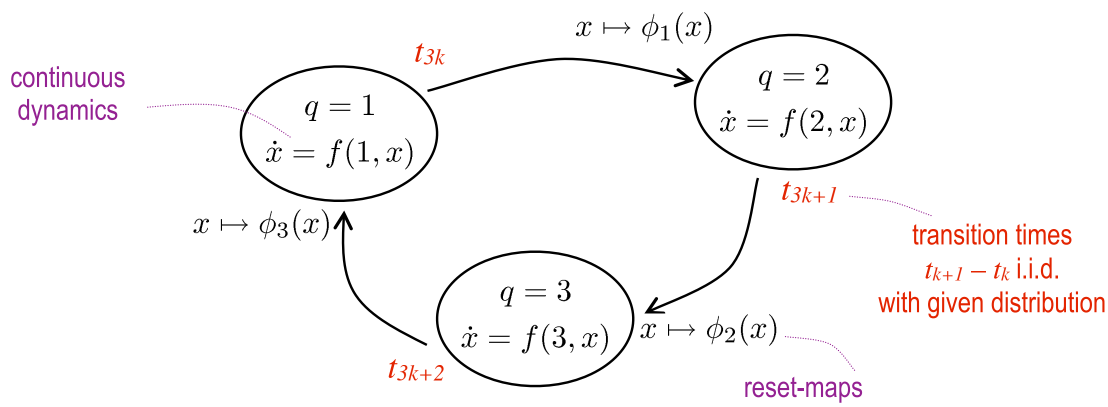


Stochastic Impulsive System

Lecture #1 Outline

- Deterministic Impulsive Systems (DISs)
- Deterministic Hybrid Systems (DHSs)
- Stochastic Hybrid Systems (SHSs)
- Simulation of SHSs
- Time-triggered SHSs

Time-triggered Stochastic Hybrid Systems



$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state

$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

$N(t) \equiv$ # of transitions before time t

renewal process
(iid inter-increment times)

(Also known as SHSs *driven by renewal processes*)

Example #4: Networked Control System

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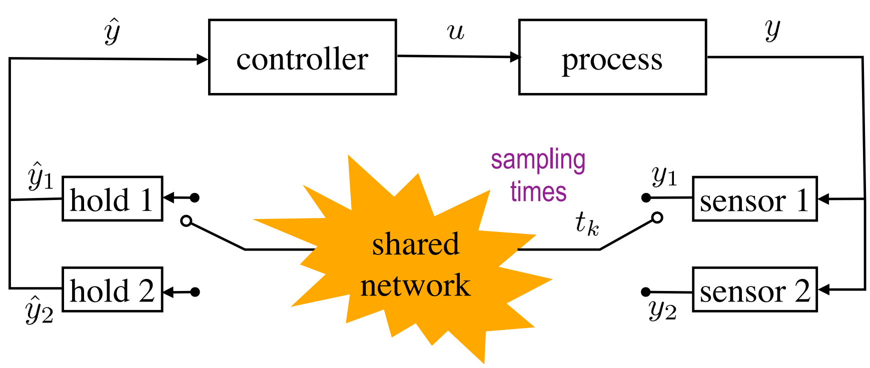


process: $\dot{x}_P = A_P x_P + C_P u$
$y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$
$\hat{y} = C_C x_C + D_C \hat{y}$

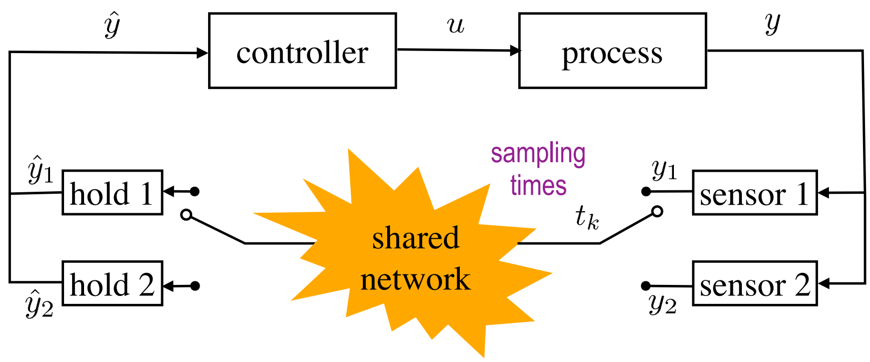
round-robin network access:

$\dot{\hat{y}} = 0$

hold

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{cases} \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix} & k \text{ odd} \\ \begin{bmatrix} \hat{y}_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} & k \text{ even} \end{cases}$$

Example #4: Networked Control System



process: $\dot{x}_P = A_P x_P + C_P u$
$y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$
$\hat{y} = C_C x_C + D_C \hat{y}$

What if the network is not available at a sample time t_k ?

1st wait until network becomes available

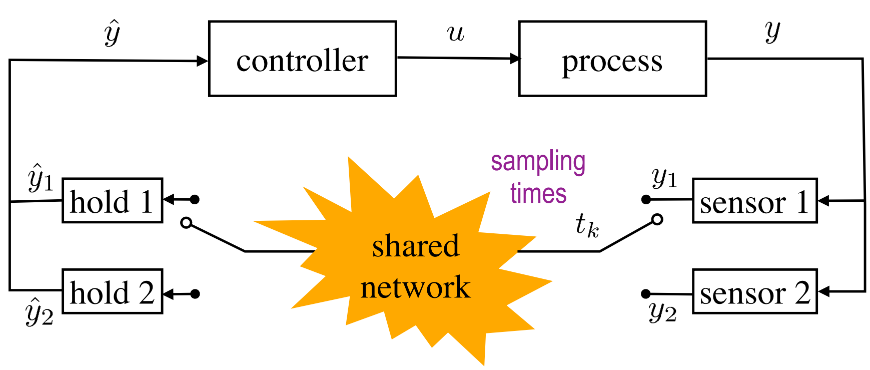
2nd send (old) data from original sampling of continuous-time output

or

2nd send (latest) data from current sampling of continuous-time output

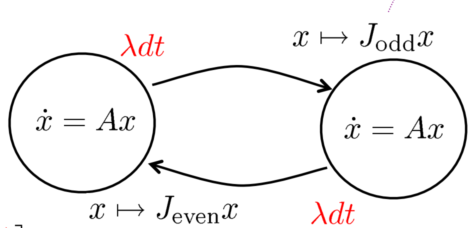
$\Rightarrow$ intersampling times $t_{k+1} - t_k$ typically become random variables

Example #4: Networked Control System



Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed

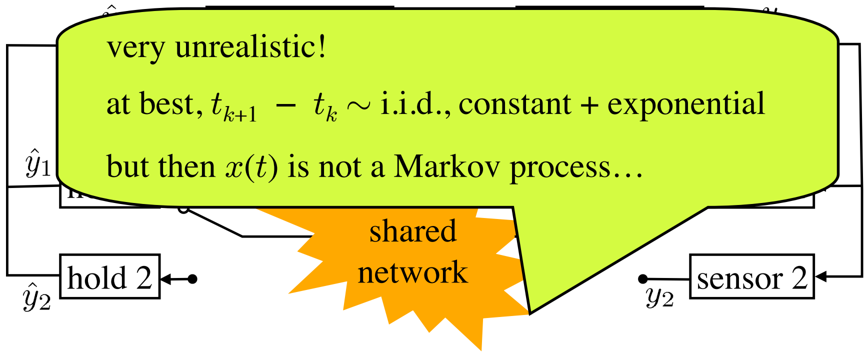
$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$



$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$

$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

Example #4: Networked Control System



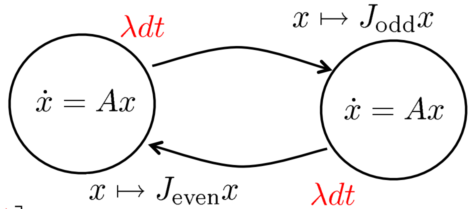
very unrealistic!

at best, $t_{k+1} - t_k \sim$ i.i.d., constant + exponential

but then $x(t)$ is not a Markov process…

Suppose $t_{k+1} - t_k \sim$ i.i.d., exponentially distributed

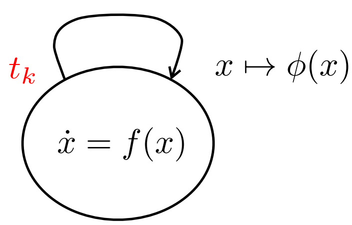
$$\begin{bmatrix}\hat{y}_1(t_k)\\ \hat{y}_2(t_k)\end{bmatrix} = \begin{bmatrix}y_1(t_k^-)\\ \hat{y}_2(t_k^-)\end{bmatrix}$$



$$x := \begin{bmatrix}x_P\\ x_C\\ \hat{y}\end{bmatrix}$$

$$\begin{bmatrix}\hat{y}_1(t_k)\\ \hat{y}_2(t_k)\end{bmatrix} = \begin{bmatrix}y_1(t_k^-)\\ \hat{y}_2(t_k^-)\end{bmatrix}$$

Time-triggered SIS



Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the t_k ?

Time-triggered SIS

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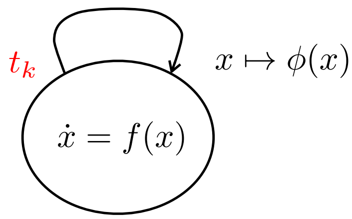

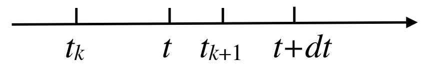

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Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the t_k ?

Recall:

$$\underbrace{\mathrm{P}\left(\text{jump in } (t, t+dt] \;\middle|\; t_k, x(t_k), \text{no jump in } [t_k, t]\right)}_{\mathrm{P}\left(t < t_{k+1} \le t + dt \;\middle|\; t_k, x(t_k), t_{k+1} > t\right)} \xrightarrow{dt \to 0} \lambda_\ell(x(t))dt$$

hazard rate

$$= \frac{F(t + dt - t_k) - F(t - t_k)}{1 - F(t - t_k)} \xrightarrow{dt \to 0} \frac{F'(t - t_k)}{1 - F(t - t_k)} dt$$

Time-triggered SIS

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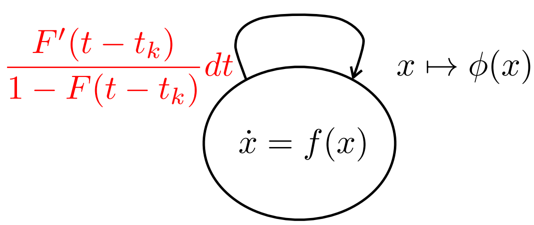

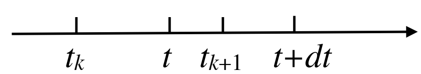

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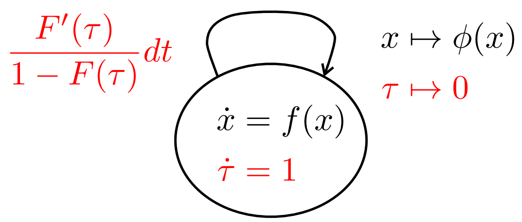
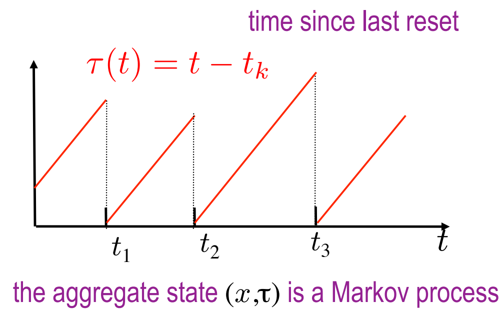
Time-triggered SIS





Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the t_k ?

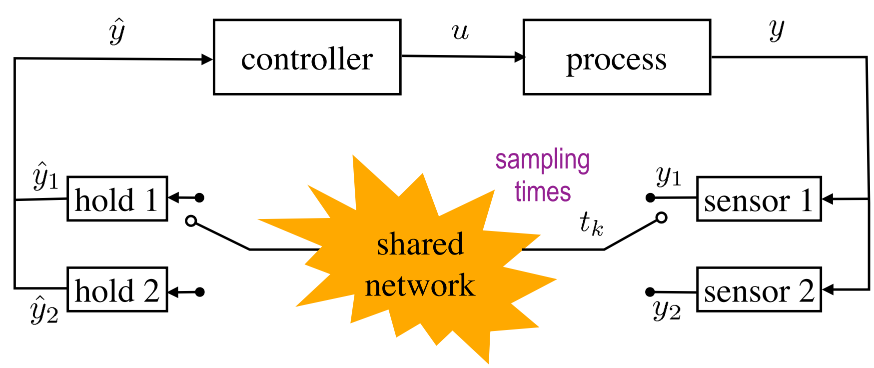
Recall:

$$\underbrace{\mathrm{P}\Big(\text{jump in } (t, t+dt] \;\Big|\; t_k, x(t_k), \text{no jump in } [t_k, t]\Big)}_{\mathrm{P}\left(t < t_{k+1} \leq t + dt \;\middle|\; t_k, x(t_k), t_{k+1} > t\right)} \xrightarrow{dt \to 0} \lambda_\ell(x(t))dt$$

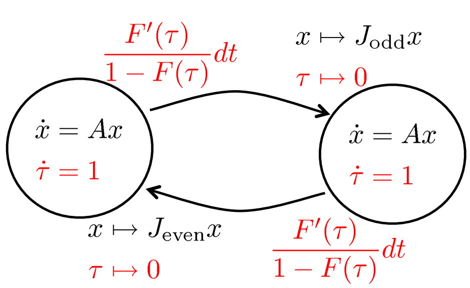
hazard rate

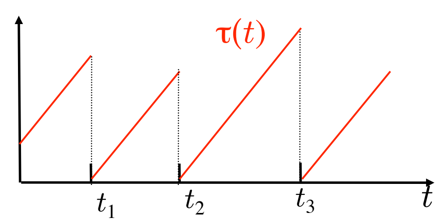
$$= \frac{F(t+dt-t_k) - F(t-t_k)}{1 - F(t-t_k)} \xrightarrow{dt\to 0} \frac{F'(t-t_k)}{1-F(t-t_k)}dt$$

Example #4: Networked Control System



Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$





$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

Example #4: Networked Control System

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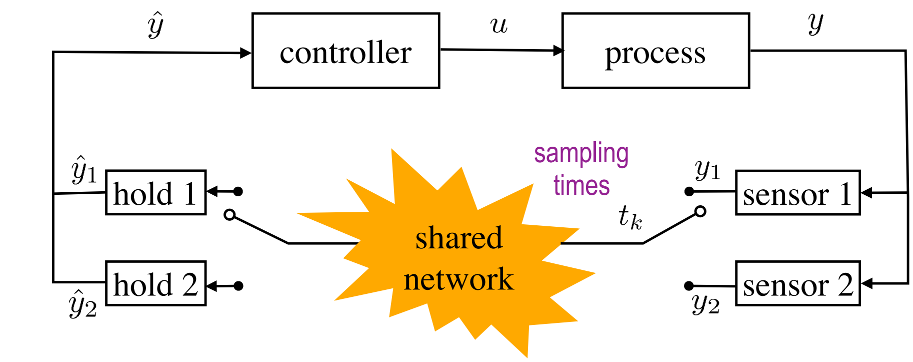
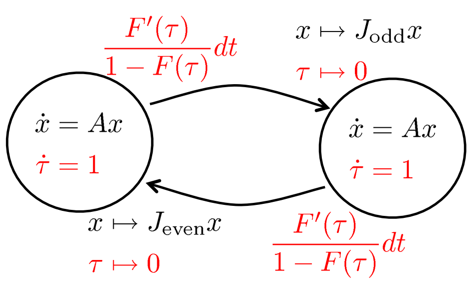
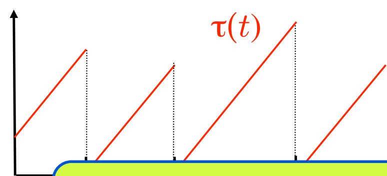


Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$





$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

This representation allows one to *combine in the same SHS* time and event triggered transitions!

Lecture #2

Analysis of Stochastic Hybrid Systems

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Center for Control, Dynamical Systems and Computation CCDC

Lecture #2 Outline

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- Infinitesimal Generator and Dynkin's Formula
- Lyapunov-based Analysis
- Stability of SHSs Driven by Renewal Processes

Main references:
Davis, "Markov Models and Optimization" Chapman & Hall,1993
Kushner, "Stochastic Stability and Control" Academic Press,1967
Antunes et al., ACC'09, CDC'09, ACC'10, CDC'10

ODE – Lie Derivative

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$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

$$\frac{dV\big(x(t)\big)}{dt} = \underbrace{\frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)}_{L_f V}$$

derivative along solution to ODE

$L_f V$
Lie derivative of V

Basis of "Lyapunov" formal arguments to establish boundedness and stability…

E.g., picking $V(x) := \|x\|^2$

$$\frac{dV\big(x(t)\big)}{dt} = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \Rightarrow \quad V\big(x(t)\big) = \|x(t)\|^2 \leq \|x(0)\|^2$$

$\|x\|^2$ remains bounded along trajectories !

ODE – Lie Derivative

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Along solutions to ODE

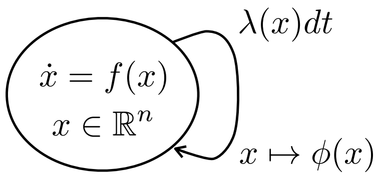
$$x(t+dt) = x(t) + \underbrace{\dot{x}(t)}_{f(x(t))}dt + O(dt^2)$$

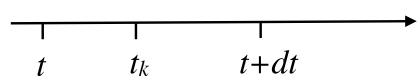
Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} V\big(x(t+dt)\big) &= V\Big(x(t) + f\big(x(t)\big)dt + O(dt^2)\Big) \\ &= V\big(x(t)\big) + \frac{\partial V\big(x(t+dt)\big)}{\partial x} f\big(x(t)\big)dt + O(dt^2) \end{aligned}$$

$$\frac{dV\big(x(t)\big)}{dt} = \lim_{dt \to 0} \frac{V\big(x(t+dt)\big) - V\big(x(t)\big)}{dt} = \frac{\partial V\big(x(t+dt)\big)}{\partial x} f\big(x(t)\big)$$

Stochastic Impulsive System





Along a sample path to the SIS

$$x(t+dt) = \begin{cases} x(t) + f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ \text{[see below]} & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump} \ldots \end{cases}$$

Assuming one jump at time $t_k \in (t, dt]$

$$x^-(t_k) = x(t) + f\big(x(t)\big)\underbrace{(t_k - t)}_{O(dt)} + O\big(\underbrace{(t_k - t)^2}_{O(dt^2)}\big)$$

continuous evolution

$$x(t_k) = \phi\big(x^-(t_k)\big) = \phi\big(x(t)\big) + \frac{\partial \phi\big(x(t)\big)}{\partial x} O(dt)$$

jump

$$x(t+dt) = x(t_k) + f\big(x(t_k)\big)\underbrace{(t + dt - t_k)}_{O(dt)} + O\big(\underbrace{(t + dt - t_k)^2}_{O(dt^2)}\big) = \phi\big(x(t)\big) + O(dt)$$

Stochastic Impulsive System

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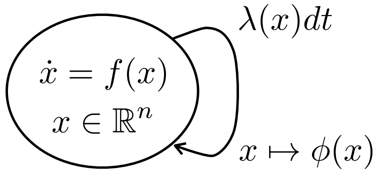

Along a sample path to the SIS

$$x(t+dt) = \begin{cases} x(t) + f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ \phi\big(x(t)\big) + O(dt) & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump } \ldots \end{cases}$$

Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

$$V\big(x(t+dt)\big) = \begin{cases} V\Big(x(t)\Big) + \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump } \ldots \end{cases}$$

Stochastic Impulsive System

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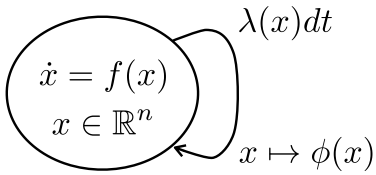
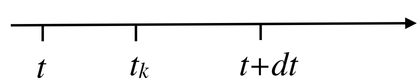
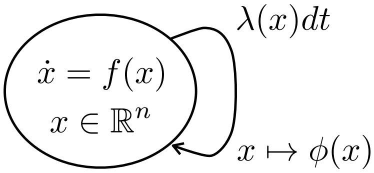


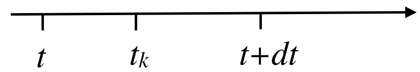


$$V\big(x(t+dt)\big) = \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump } \ldots \end{cases}$$

$$= \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{w.p. } 1 - \lambda\big(x(t)\big)dt \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{w.p. } \lambda\big(x(t)\big)dt \\ ??? & \text{w.p. } O(dt^2) \end{cases}$$

Stochastic Impulsive System





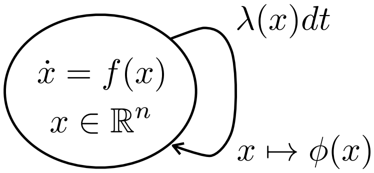
$$V\big(x(t+dt)\big) = \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump ...} \end{cases}$$

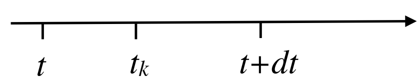
$$= \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{w.p. } 1 - \lambda\big(x(t)\big)dt \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{w.p. } \lambda\big(x(t)\big)dt \\ ??? & \text{w.p. } O(dt^2) \end{cases}$$

Given $x(t)$

$$\mathrm{E}\Big[V\big(x(t+dt)\big) \mid x(t)\Big] = \Big(V\big(x(t)\big) + \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)dt + O(dt^2)\Big)\Big(1 - \lambda\big(x(t)\big)dt\Big) + V\Big(\phi\big(x(t)\big)\Big)\lambda\big(x(t)\big)dt + O(dt^2)$$

Stochastic Impulsive System





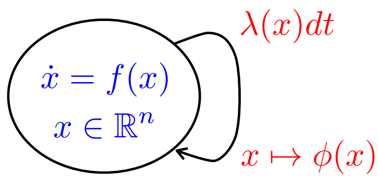
$$V\big(x(t+dt)\big) = \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{no jumps in } (t, dt] \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{one jump in } (t, dt] \\ ??? & \text{more than one jump ...} \end{cases}$$

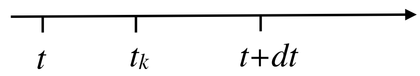
$$= \begin{cases} V\big(x(t)\big) + \frac{\partial V(x(t))}{\partial x} f\big(x(t)\big)dt + O(dt^2) & \text{w.p. } 1 - \lambda\big(x(t)\big)dt \\ V\Big(\phi\big(x(t)\big)\Big) + O(dt) & \text{w.p. } \lambda\big(x(t)\big)dt \\ ??? & \text{w.p. } O(dt^2) \end{cases}$$

Given $x(t)$

$$\mathrm{E}\Big[V\big(x(t+dt)\big) \mid x(t)\Big] = V\big(x(t)\big) + \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)\color{red}{dt} - V\big(x(t)\big)\lambda\big(x(t)\big)\color{red}{dt} + V\Big(\phi\big(x(t)\big)\Big)\lambda\big(x(t)\big)\color{red}{dt} + O(\color{red}{dt^2})$$

Stochastic Impulsive System





$$\frac{d\,\mathrm{E}\left[V\big(x(\tau)\big)\mid x(t)\right]}{d\tau}\Big|_{\tau=t} = \lim_{dt\to 0}\frac{\mathrm{E}\left[V\big(x(t+dt)\big)-V\big(x(t)\big)\mid x(t)\right]}{dt}$$

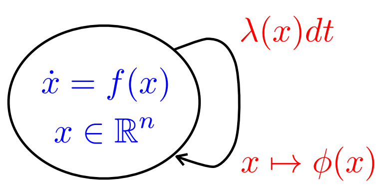
$$= \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big) + \Big(V\Big(\phi\big(x(t)\big)\Big) - V\big(x(t)\big)\Big)\lambda\big(x(t)\big)$$

(implicit assumption that terms O(dt^2) do not cause trouble…
can be overcome by working with (bounded) stopped versions of the process)

Given $x(t)$

$$\mathrm{E}\left[V\big(x(t+dt)\big)\mid x(t)\right] = V\big(x(t)\big) + \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)dt - V\big(x(t)\big)\lambda\big(x(t)\big)dt$$
$$+ V\Big(\phi\big(x(t)\big)\Big)\lambda\big(x(t)\big)dt + O(dt^2)$$

Generator of a Stochastic Impulsive System



Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

x is discontinuous, but the expected value is differentiable

$$\frac{d}{dt}\mathrm{E}\left[V\big(x(t)\big)\right] = E\left[(LV)\big(x(t)\big)\right]$$

Dynkin's formula (in differential form)

where

Lie derivative

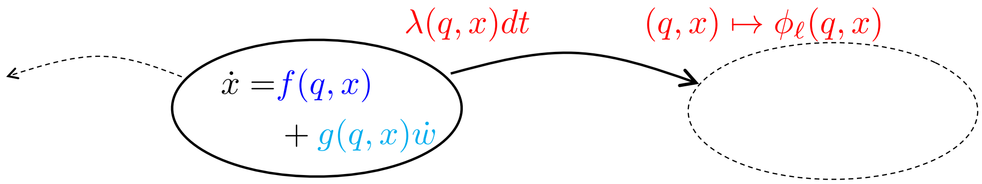
Reset term (absent for deterministic ODEs)

$$(LV)(x) = \frac{\partial V(x)}{\partial x} f(x) + \underbrace{\Big(V\big(\phi(x)\big) - V(x)\Big)}_{\text{instantaneous variation}}\lambda(x)$$

intensity

(extended) generator of the SIS

Generator of a Stochastic Hybrid System



Given scalar-valued function $V : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}$

x & q are discontinuous, but the expected value is differentiable

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(q(t),x(t)\big)\Big] = E\Big[(LV)\big(q(t),x(t)\big)\Big]$$

Dynkin's formula (in differential form)

where

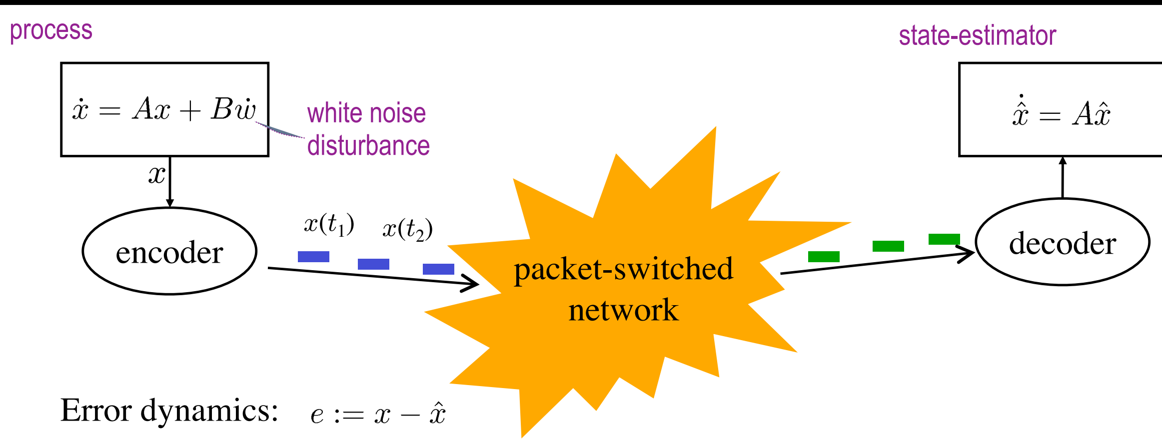
(extended) generator of the SHS

$$(LV)(q,x) := \frac{\partial V}{\partial x}(q,x) f(q,x)$$ Lie derivative

$$+ \sum_{\ell=1}^{m} \underbrace{\lambda(q,x)}_{\text{intensity}} \Big(\overbrace{V\big(\phi_\ell(q,x)\big) - V(q,x)}^{\text{instantaneous variation}} \Big)$$ Reset term

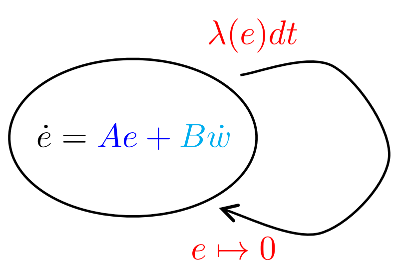
$$+ \frac{1}{2}\,\mathrm{trace}\left(g(q,x)' \frac{\partial^2 V}{\partial x^2} g(q,x) \right)$$ Diffusion term

Example #3: Remote estimation



Error dynamics: $e := x - \hat{x}$

prob. of sending data in $[t,t+dt)$ depends on current error e



reset error to zero

$$(LV)(e) := \frac{\partial V}{\partial e}(e) Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\left(B' \frac{\partial^2 V}{\partial e^2}(e) B\right)$$

Lecture #2 Outline

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- Infinitesimal Generator and Dynkin's Formula
- Lyapunov-based Analysis
- Stability of SHSs Driven by Renewal Processes

Lyapunov Analysis – ODEs

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$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

$$\frac{dV\big(x(t)\big)}{dt} = \frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)$$

Suppose $\begin{cases} V(x) \geq 0 \\ \frac{\partial V(x)}{\partial x} f(x) \leq 0 \end{cases} \quad \forall x$

Then $\displaystyle \frac{dV\big(x(t)\big)}{dt} = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \Rightarrow \quad V\big(x(t)\big) \leq V\big(x_0\big) \quad \forall t \geq 0$

zero at zero & monotone increasing

"Squeezing" $V(x)$ between two class-K functions $\quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

$$\|x(t)\| \leq \alpha_1^{-1}\big(\alpha_2(\|x_0\|)\big) \quad \forall t \geq 0$$

$\|x(t)\|$ can be kept arbitrarily small by making $\|x_0\|$ small

Lyapunov Analysis – SISs

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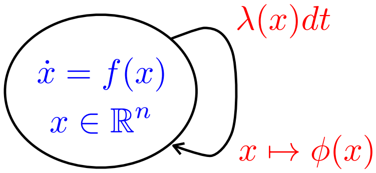

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose $\begin{cases} V(x) \geq 0 \\ LV(x) \leq 0 \end{cases} \quad \forall x$

Pick $T, K > 0$ and define

$$\tau^* := \begin{cases} T & V\big(x(t)\big) < K, \forall t \in [0,T] \\ \text{1st time } V\big(x(t)\big) \geq K & \text{otherwise} \end{cases}$$

$$z^* := \begin{cases} 0 & V\big(x(t)\big) < K, \forall t \in [0,T] \\ 1 & \text{otherwise} \end{cases}$$

From Dynkin's formula

$$\mathrm{E}\Big[V\big(x(\tau^*)\big)\Big] \leq \mathrm{E}\Big[V\big(x(0)\big)\Big] = V(x_0)$$

Lyapunov Analysis – SISs

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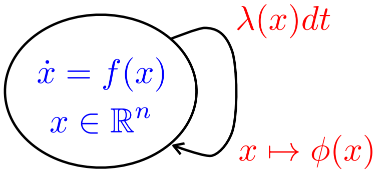

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose $\begin{cases} V(x) \geq 0 \\ LV(x) \leq 0 \end{cases} \quad \forall x$

Pick $T, K > 0$ and define

$$\tau^* := \begin{cases} T & V\big(x(t)\big) < K, \forall t \in [0,T] \\ \text{1st time } V\big(x(t)\big) \geq K & \text{otherwise} \end{cases}$$

$$z^* := \begin{cases} 0 & V\big(x(t)\big) < K, \forall t \in [0,T] \\ 1 & \text{otherwise} \end{cases}$$

From Dynkin's formula

$$\mathrm{E}\Big[\underbrace{V\big(x(\tau^*)\big)}_{z^*V\big(x(\tau^*)\big) + \underbrace{(1-z^*)V\big(x(\tau^*)\big)}_{\geq 0} \geq z^*K}\Big] \leq \mathrm{E}\Big[V\big(x(0)\big)\Big] = V(x_0) \quad \Rightarrow \quad KE[z^*] \leq V(x_0)$$

Lyapunov Analysis – SISs

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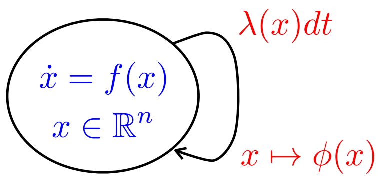

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose $\begin{cases} V(x) \geq 0 \\ LV(x) \leq 0 \end{cases} \quad \forall x$

Pick $T, K > 0$ and define

$$\tau^* := \begin{cases} T & V\big(x(t)\big) < K, \forall t \in [0,T] \\ \text{1st time } V\big(x(t)\big) \geq K & \text{otherwise} \end{cases}$$

$$z^* := \begin{cases} 0 & V\big(x(t)\big) < K, \forall t \in [0,T] \\ 1 & \text{otherwise} \end{cases}$$

From Dynkin's formula

$$\mathrm{E}\Big[\underbrace{V\big(x(\tau^*)\big)}\Big] \leq \mathrm{E}\Big[V\big(x(0)\big)\Big] = V(x_0)$$

$$z^* V\big(x(\tau^*)\big) + \underbrace{(1 - z^*) V\big(x(\tau^*)\big)}_{\geq 0} \geq z^* K$$

$$\Rightarrow \quad K E[z^*] \leq V(x_0)$$

$$\mathrm{P}\Big(V\big(x(t)\big) \text{ ever becomes} \geq K\Big)$$

Lyapunov Stability in Probability

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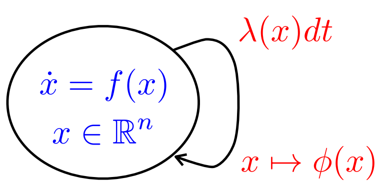


$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose

Doob's (Martingale) inequality

$$\begin{cases} V(x) \geq 0 \\ LV(x) \leq 0 \end{cases} \quad \forall x \quad \Rightarrow \quad \mathrm{P}\Big(V\big(x(t)\big) \text{ ever becomes} \geq K\Big) \leq \frac{V(x_0)}{K}$$

zero at zero & monotone increasing

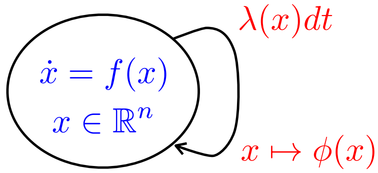
"Squeezing" $V(x)$ between two class-K functions $\quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

$$\mathrm{P}\Big(\|x(t)\| \text{ ever becomes} \geq M\Big) \leq \frac{\alpha_2(\|x_0\|)}{\alpha_1(M)}$$

Lyapunov stability in probability

Probability of $\|x(t)\|$ exceeding any given bound M, can be made arbitrarily small by making $\|x_0\|$ small

Almost Sure Asymptotic Stability



$$\frac{d}{dt}\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose

zero at zero & monotone increasing

$$\begin{cases} \alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \\ LV(x) \le -\alpha_3(\|x\|) \end{cases}$$

Then

$$\mathrm{P}\Big(\|x(t)\| \text{ ever becomes} \ge M\Big) \le \frac{\alpha_2(\|x_0\|)}{\alpha_1(M)}$$

almost sure (a.s.) asymptotic stability

$$\mathrm{P}\,\big(x(t) \to 0\big) = 1$$

Proof also follows from Dynkin's formula

Almost Sure Asymptotic Stability

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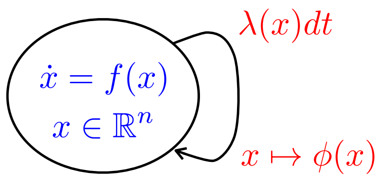

$$\frac{d}{dt}\mathrm{E}\Big[V\big(x(t)\big)\Big] = E\Big[(LV)\big(x(t)\big)\Big]$$

Suppose

zero at zero & monotone increasing

$$\begin{cases} \alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \\ LV(x) \le -\alpha_3(\|x\|) \end{cases}$$

Then

$$\mathrm{P}\Big(\|x(t)\| \text{ ever becomes} \ge M\Big) \le \frac{\alpha_2(\|x_0\|)}{\alpha_1(M)}$$

almost sure (a.s.) asymptotic stability

$$\mathrm{P}\,\big(x(t) \to 0\big) = 1$$

Stability in probability & a.s. asymptotic stability are *sample-path properties*
(bound the probabilities of ill-behaved paths)

Ensemble Notions of Stability

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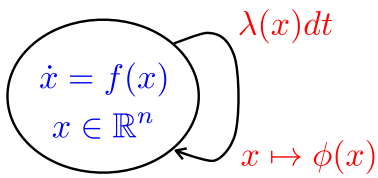

$$\frac{d}{dt}\,\mathrm{E}\left[V\big(x(t)\big)\right] = E\left[(LV)\big(x(t)\big)\right]$$

Suppose $\begin{cases} V(x) \geq 0 \\ LV(x) \leq -W(x) \end{cases}$

Integrating Dynkin's formula

$$\underbrace{\mathrm{E}\left[V\big(x(T)\big)\right]}_{\geq 0} - V(x_0) \leq -\int_0^T \mathrm{E}\left[W\big(x(t)\big)\right]dt \quad \forall T > 0$$

$$\Rightarrow \quad \int_0^T \mathrm{E}\left[W\big(x(t)\big)\right]dt \leq V(x_0)$$

$$\int_0^\infty \mathrm{E}\left[W\big(x(t)\big)\right]dt < \infty$$

stochastic stability
(mean square if
$W(x)=\|x\|^2$)

Ensemble Notions of Stability

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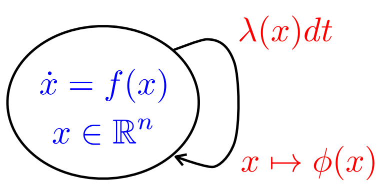

$$\frac{d}{dt}\,\mathrm{E}\left[V\big(x(t)\big)\right] = E\left[(LV)\big(x(t)\big)\right]$$

Suppose $\begin{cases} V(x) \geq W(x) \geq 0 \\ LV(x) \leq -\mu V + c \end{cases}$

From Dynkin's formula

$$\frac{d}{dt}\,\mathrm{E}\left[V\big(x(t)\big)\right] \leq -\mu E\left[V\big(x(t)\big)\right] + c$$

$$\Rightarrow \quad \mathrm{E}\left[W\big(x(t)\big)\right] \leq \mathrm{E}\left[V\big(x(t)\big)\right] \leq e^{-\mu t}V(x_0) + \frac{c}{\mu}$$

exponential stability
(mean square if
$W(x)=\|x\|^2$)

Example #3: Remote estimation

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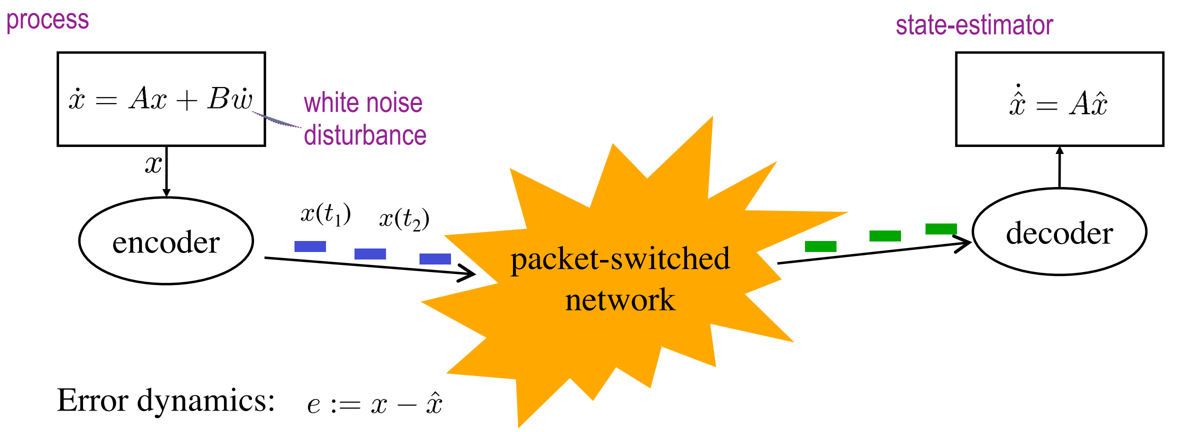
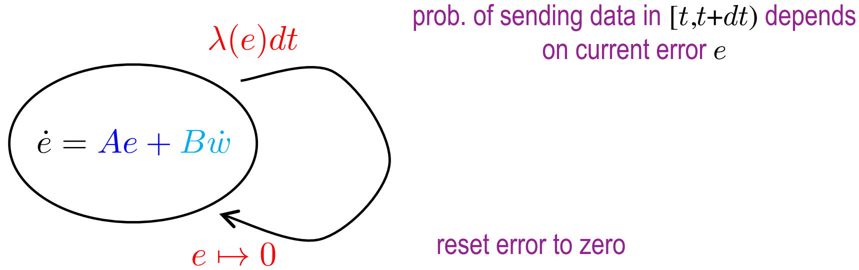


Error dynamics: $e := x - \hat{x}$



$$(LV)(e) := \frac{\partial V}{\partial e}(e)Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2 V}{\partial e^2}(e)B\right)$$

Lyapunov-based stability analysis

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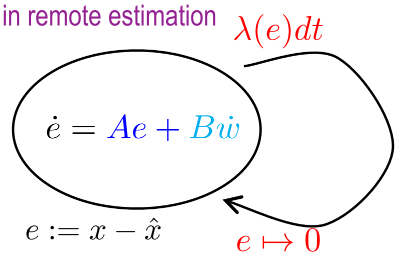
error dynamics in remote estimation



Dynkin's formula

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(e(t)\big)\Big] = E\Big[(LV)\big(e(t)\big)\Big]$$

$$(LV)(e) := \frac{\partial V}{\partial e}Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2 V}{\partial e^2}B\right)$$

2nd moment of the error:

$$V(e) = e'Pe \;\Rightarrow\; (LV)(e) = e'\left[\left(A - \frac{\lambda(e)}{2}I\right)'P + P\left(A - \frac{\lambda(e)}{2}I\right)\right]e + \mathrm{trace}\, B'PB$$

For constant rate: $\lambda(e) = \gamma$

$$A - \frac{\gamma}{2}I \text{ Hurwitz} \quad\Rightarrow\quad \exists \mu > 0, P \geq I : \left(A - \frac{\gamma}{2}I\right)'P + P\left(A - \frac{\gamma}{2}I\right) \leq -\mu P$$

Lyapunov-based stability analysis

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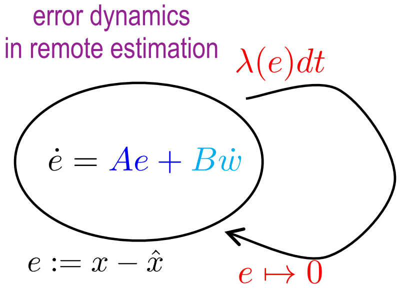

Dynkin's formula

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$$(LV)(e) := \frac{\partial V}{\partial e}Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2 V}{\partial e^2}B\right)$$

2nd moment of the error:

$$V(e) = e'Pe \;\Rightarrow\quad (LV)(e) = e'\left[\left(A - \frac{\lambda(e)}{2}I\right)'P + P\left(A - \frac{\lambda(e)}{2}I\right)\right]e + \mathrm{trace}\,B'PB$$

For constant rate: $\lambda(e) = \gamma$

$$A - \frac{\gamma}{2}I \text{ Hurwitz} \quad \Rightarrow \quad \exists \mu > 0, P \geq I : \left(A - \frac{\gamma}{2}I\right)'P + P\left(A - \frac{\gamma}{2}I\right) \leq -\mu P$$

$$\begin{cases} V(e) \geq \|e\|^2 \geq 0 \\ LV(e) \leq -\mu V + \mathrm{trace}\,B'PB \end{cases} \quad \Rightarrow \quad \mathrm{E}\left[\|e(t)\|^2\right] \leq e^{-\mu t} e_0' P e_0 + \frac{\mathrm{trace}\,B'PB}{\mu}$$

Lyapunov-based stability analysis

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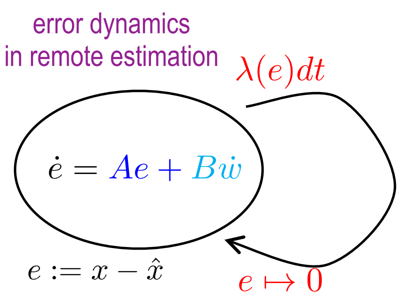

Dynkin's formula

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(e(t)\big)\Big] = E\Big[(LV)\big(e(t)\big)\Big]$$

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2nd moment of the error:

$$V(e) = e'Pe \;\Rightarrow\quad (LV)(e) = e'\left[\left(A - \frac{\lambda(e)}{2}I\right)'P + P\left(A - \frac{\lambda(e)}{2}I\right)\right]e + \mathrm{trace}\,B'PB$$

For radially unbounded rate: $\lambda(e)$

$$V(e) = \|e\|^2 \quad \Rightarrow \quad \underbrace{(LV)(e) + \mu V = 2e'Ae + \underbrace{\mu\|e\|^2 - \lambda(e)\|e\|^2}_{\substack{\downarrow \\ -\infty}\ \text{as } \|e\| \to \infty} + \mathrm{trace}\,B'PB}_{\forall \mu,\ \text{must be upper bounded by some } c < \infty}$$

Lyapunov-based stability analysis

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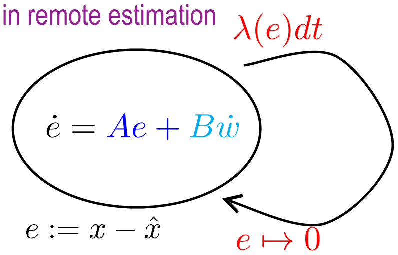
error dynamics
in remote estimation



Dynkin's formula

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(e(t)\big)\Big] = E\Big[(LV)\big(e(t)\big)\Big]$$

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2nd moment of the error:

$$V(e) = e'Pe \;\Rightarrow\quad (LV)(e) = e'\Big[\Big(A - \frac{\lambda(e)}{2}I\Big)'P + P\Big(A - \frac{\lambda(e)}{2}I\Big)\Big]e + \mathrm{trace}\,B'PB$$

For radially unbounded rate: $\lambda(e)$

$$V(e) = \|e\|^2 \quad\Rightarrow\quad \forall \mu > 0\;\exists c < \infty:\; (LV)(e) + \mu V \le c$$

$$\begin{cases} V(e) \ge \|x\|^2 \ge 0 \\ LV(e) \le -\mu V + c \end{cases} \quad\Rightarrow\quad \mathrm{E}\big[\|e(t)\|^2\big] \le e^{-\mu t} e_0' P e_0 + \frac{c}{\mu}$$

Mean-square exp. stability, regardless of how unstable A is (true for every moment)

Lyapunov-based stability analysis

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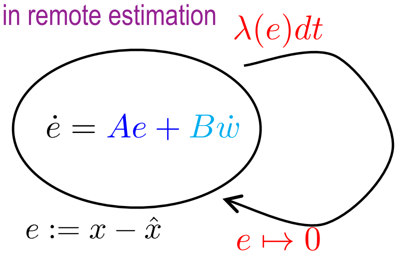
error dynamics
in remote estimation



Dynkin's formula

$$\frac{d}{dt}\,\mathrm{E}\Big[V\big(e(t)\big)\Big] = E\Big[(LV)\big(e(t)\big)\Big]$$

$$(LV)(e) := \frac{\partial V}{\partial e}Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\Big(B'\frac{\partial^2 V}{\partial e^2}B\Big)$$

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $\mathrm{E}[\,e\,] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\,\|e\|^m\,]$ bounded if and only if $\gamma > m\,\Re[\lambda(A)]$

getting more moments bounded requires higher comm. rates

For radially unbounded rate: $\lambda(e)$ (reactive transmissions)

5. $\mathrm{E}[\,e\,] \to 0$ (always)
6. $\mathrm{E}[\,\|e\|^m\,]$ bounded $\forall\, m$

Moreover, one can achieve the same $\mathrm{E}[\,\|e\|^2\,]$ with less communication than with a constant rate or periodic transmissions…

Lecture #2 Outline

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- Infinitesimal Generator and Dynkin's Formula
- Lyapunov-based Analysis
- Stability of SHSs Driven by Renewal Processes

Time-triggered Linear SIS

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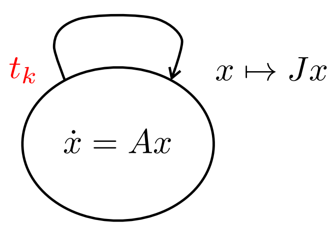

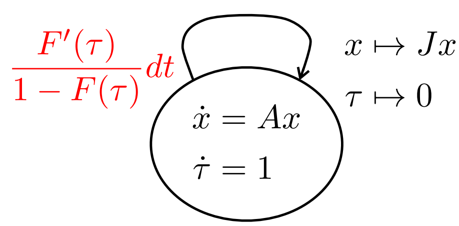

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Defining $x_k := x(t_k)$ state at jump times

$$x_{k+1} = Je^{A(t_{k+1}-t_k)}x_k$$

reset continuous evolution

Time-triggered Linear SIS

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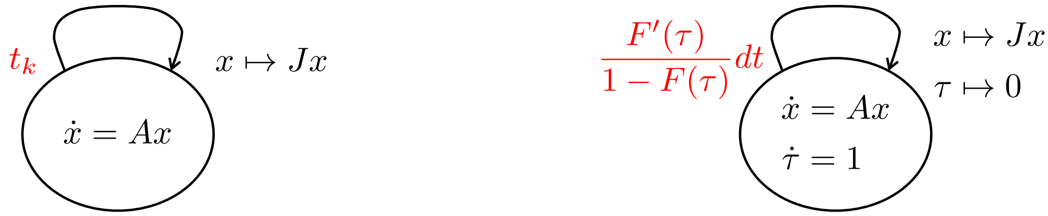

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Defining $x_k := x(t_k)$ state at jump times

$$x_{k+1} = J e^{A(t_{k+1} - t_k)} x_k$$

reset

continuous evolution

For a given symmetric matrix P

$$\mathrm{E}[x'_{k+1} P x_{k+1} \mid x_k] = \mathrm{E}\left[x'_k e^{A'(t_{k+1}-t_k)} J' P J e^{A(t_{k+1}-t_k)} x_k \mid x_k\right]$$

$$= x'_k \, \mathrm{E}\left[\underbrace{e^{A'(t_{k+1}-t_k)} J' P J e^{A(t_{k+1}-t_k)}}_{}\right] x_k$$

expectation with respect $t_{k+1} - t_k$
(i.i.d., with cumulative distribution function F)

Time-triggered Linear SIS

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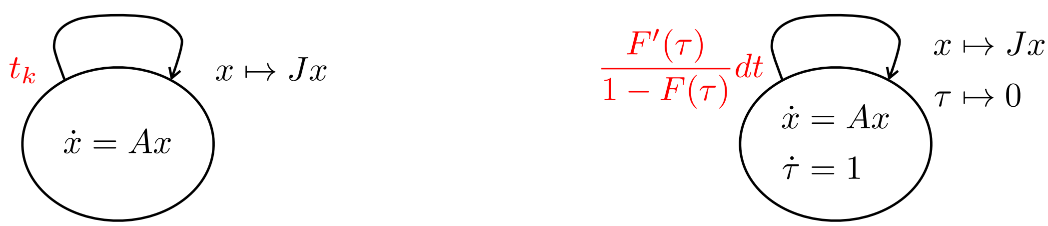

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Defining $x_k := x(t_k)$ state at jump times

$$x_{k+1} = J e^{A(t_{k+1} - t_k)} x_k$$

reset

continuous evolution

For a given symmetric matrix P

$$\mathrm{E}[x'_{k+1} P x_{k+1} \mid x_k] = \mathrm{E}\left[x'_k e^{A'(t_{k+1}-t_k)} P e^{A(t_{k+1}-t_k)} x_k \mid x_k\right]$$

$$= x'_k \, \mathrm{E}\left[e^{A'(t_{k+1}-t_k)} P e^{A(t_{k+1}-t_k)}\right] x_k$$

Suppose

$$\mathrm{E}_{F(s)}\left[e^{A's} P e^{As}\right] \le \gamma P, \ \gamma < 1 \quad \Rightarrow \quad \mathrm{E}[x'_{k+1} P x_{k+1}] \le \gamma \, \mathrm{E}[x'_k P x_k]$$

Time-triggered Linear SIS

UC SANTA BARBARA engineering

t_k $F'(\tau)$ $x \mapsto Jx$ $\tau \mapsto 0$

If there exists

$$P > 0, \quad \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$$

then

$$\lim_{k\to\infty} \|x_k\| = 0 \quad \text{(exponentially fast)}$$

What about $x(t)$ between jumps?

Defining x

reset

continuous evolution

For a given symmetric matrix P

$$\begin{aligned}\mathrm{E}[x'_{k+1}Px_{k+1} \mid x_k] &= \mathrm{E}\left[x'_k e^{A'(t_{k+1}-t_k)}Pe^{A(t_{k+1}-t_k)}x_k \mid x_k\right] \\ &= x'_k\,\mathrm{E}\left[e^{A'(t_{k+1}-t_k)}Pe^{A(t_{k+1}-t_k)}\right]x_k\end{aligned}$$

Suppose

$$\mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] \le \gamma P,\ \gamma < 1 \quad \Rightarrow \quad \mathrm{E}[x'_{k+1}Px_{k+1}] \le \gamma\,\mathrm{E}[x'_kPx_k]$$

Time-triggered Linear SIS

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t_k $x \mapsto Jx$

$\dot{x} = Ax$

$\dfrac{F'(\tau)}{1-F(\tau)}dt$ $x \mapsto Jx$ $\tau \mapsto 0$

$\dot{x} = Ax$

$\dot{\tau} = 1$

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Theorem:

system is mean-square stochastically stable, i.e., $\displaystyle\int_0^\infty \mathrm{E}[\|x(t)\|^2]dt < \infty$

$\Updownarrow$

LMI on $P_{n\times n}$

$$\mathrm{E}_{F(s)}\left[\int_0^s e^{A't}e^{At}dt\right] < \infty \quad \text{and} \quad \exists P > 0 : \mathrm{E}_{F(s)}\left[e^{A's}J'PJe^{As}\right] - P < 0$$

or

spectral radius condition on $n^2\times\ n^2$ matrix

$$\sigma\Big(\mathrm{E}_{F(s)}\left[e^{A's}J' \otimes e^{A's}J'\right]\Big) < 1$$

expected value w.r.t. inter-jump times

Kronecker product

Time-triggered Linear SIS

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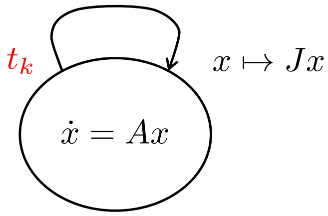

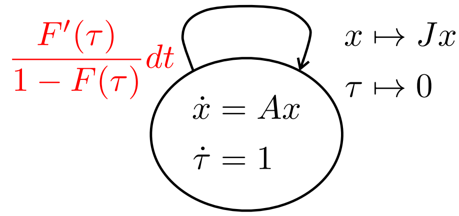

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Theorem:

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square stochastic stability

  $\& \ \mathrm{E}_{F(s)}[e^{A's}e^{As}] = \int_0^\infty e^{A's}e^{As}F(ds) < \infty \quad \Leftrightarrow \quad \int_0^\infty \mathrm{E}[\|x(t)\|^2]dt < \infty$

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square asymptotic stability

  $\& \ \lim_{s\to\infty} e^{A's}e^{As}\big(1-F(s)\big) = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] = 0$

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square exponential stability

  $\& \ \lim_{s\to\infty} e^{A's}e^{As}\big(1-F(s)\big) \overset{\text{exp. fast}}{=} 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] \overset{\text{exp. fast}}{=} 0$

Time-triggered Linear SIS

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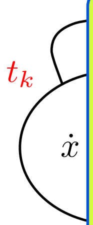

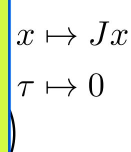

All stability notions require

$$\lim_{k\to\infty} \|x_k\| = 0 \quad \text{exponentially fast}$$

the conditions essentially only differ on the requirements on the tail of distribution

$$1 - F(s) = \mathrm{P}(t_{k+1} - t_k > s)$$

Theorem:

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square stochastic stability

  $\& \ \mathrm{E}_{F(s)}[e^{A's}e^{As}] = \int_0^\infty e^{A's}e^{As}F(ds) < \infty \quad \Leftrightarrow \quad \int_0^\infty \mathrm{E}[\|x(t)\|^2]dt < \infty$

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square asymptotic stability

  $\& \ \lim_{s\to\infty} e^{A's}e^{As}\big(1-F(s)\big) = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] = 0$

- $P > 0, \mathrm{E}_{F(s)}\left[e^{A's}Pe^{As}\right] < P$ — mean-square exponential stability

  $\& \ \lim_{s\to\infty} e^{A's}e^{As}\big(1-F(s)\big) \overset{\text{exp. fast}}{=} 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] \overset{\text{exp. fast}}{=} 0$

Example #4: Networked Control System

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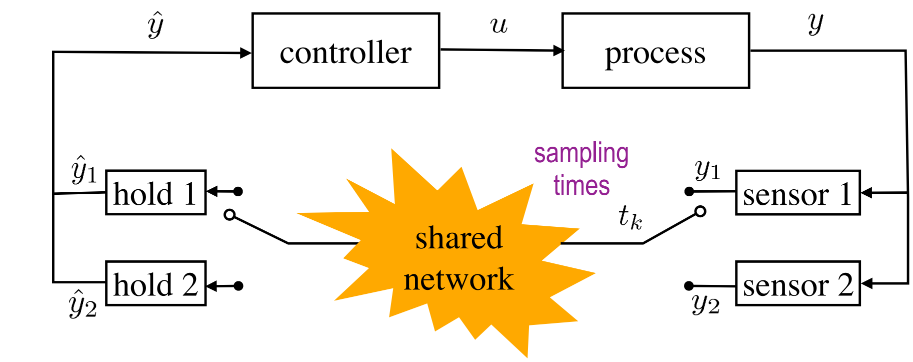
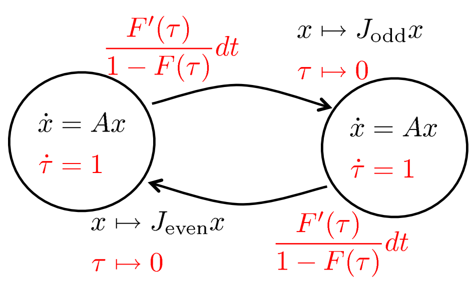
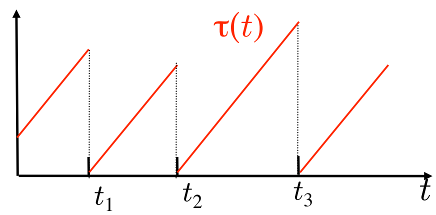


Suppose $t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$





$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

Previous results (extended to SHSs) provide nec. & suff. stability conditions when process and controller are linear

Time-triggered Linear SIS

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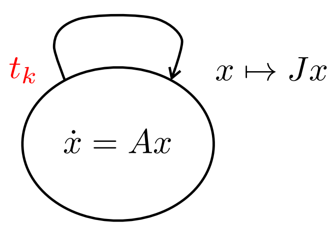

$$\frac{F'(\tau)}{1-F(\tau)}dt \qquad \begin{aligned} x &\mapsto Jx \\ \tau &\mapsto 0 \end{aligned}$$

$$\dot{x} = Ax \qquad \dot{\tau} = 1$$

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Theorem:

system is mean exponentially stable, i.e., $\quad \mathrm{E}[\|x(t)\|^2] \le ce^{-\alpha t}\|x(0)\|^2, \quad \forall t \ge 0$

$$\Updownarrow$$

$\exists P(\tau)$ such that defining $V(x,\tau) := x'P(\tau)x$

Lyapunov-like function
quadratic on x for fixed τ

$$\begin{cases} c_1 I \le P(\tau) \le c_2 I & \Rightarrow \quad V \text{ is positive definite} \\ (LV)(x,\tau) \le -\epsilon V(x,\tau) & \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[V(x,\tau)] \le -\epsilon\,\mathrm{E}[V(x,\tau)] \end{cases}$$

(essentially a converse Lyapunov stability result)

Time-triggered Linear SIS

t_k

$\dot{x} = A$…

$x \mapsto Jx$

$\tau \mapsto 0$

Motivates the use of Lyapunov functions of the form

$$V(x,\tau) := \gamma(\tau) W(x)$$

for nonlinear systems.

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Theorem:

system is mean exponentially stable, i.e., $\mathrm{E}[\|x(t)\|^2] \leq c e^{-\alpha t} \|x(0)\|^2, \quad \forall t \geq 0$

$$\Updownarrow$$

$\exists P(\tau)$ such that defining $V(x,\tau) := x' P(\tau) x$

Lyapunov-like function quadratic on x for fixed τ

$$\begin{cases} c_1 I \leq P(\tau) \leq c_2 I & \Rightarrow \quad V \text{ is positive definite} \\ (LV)(x,\tau) \leq -\epsilon V(x,\tau) & \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}[V(x,\tau)] \leq -\epsilon\, \mathrm{E}[V(x,\tau)] \end{cases}$$

(essentially a converse Lyapunov stability result)

NCS Protocol Design

Supplemental material

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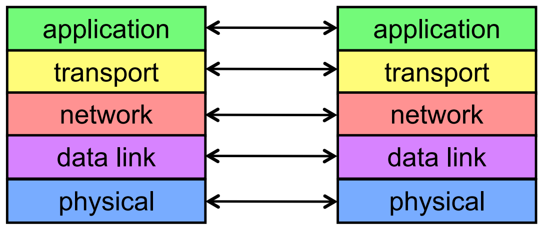
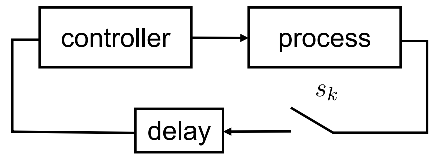
Network protocols & Control laws

network view:



control view:



This lecture: ***Co-design of network protocols and control algorithms***

1. Characterize *key parameters* that determine the stability/performance of a networked controls system
2. Construct *protocols* that directly take these parameters into considerations

Illustrative examples:

- data link layer: medium access control
- transport layer: error correction (& flow control)
- network layer: routing

Control Area Network

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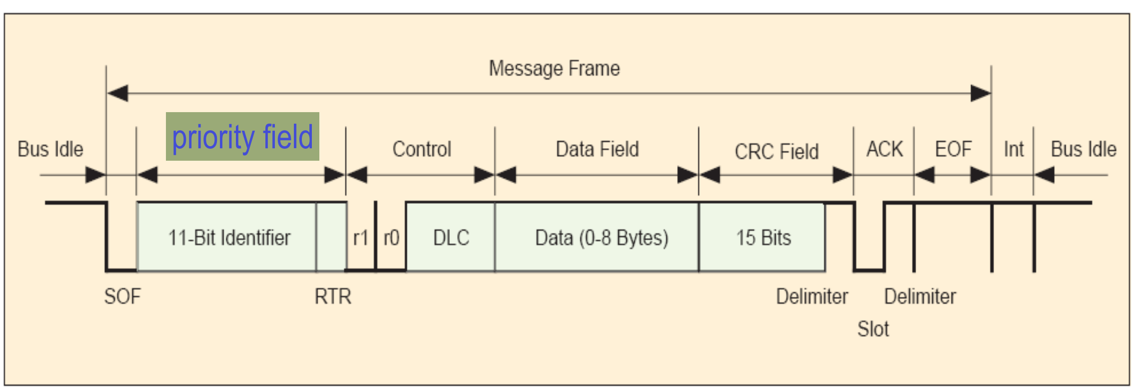
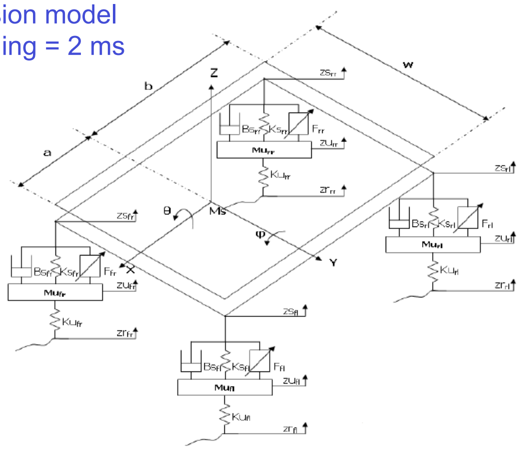
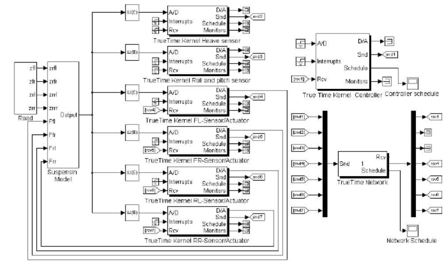
- serial communication, wired bus standard
- designed for automation systems: passenger cars, trucks, boats, spacecrafts, printers
- short messages for time critical applications
- collision-free, priority-based medium access:
  - highest priority message gains access to network
  - lower priority messages back off and wait



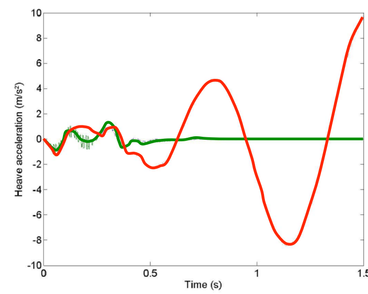
Message scheduling - does priority matter?

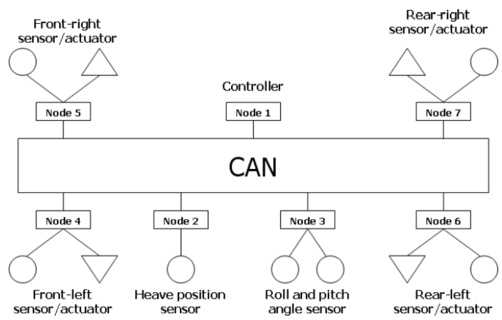
Active suspension model
constant sampling = 2 ms





Simulated with TrueTime





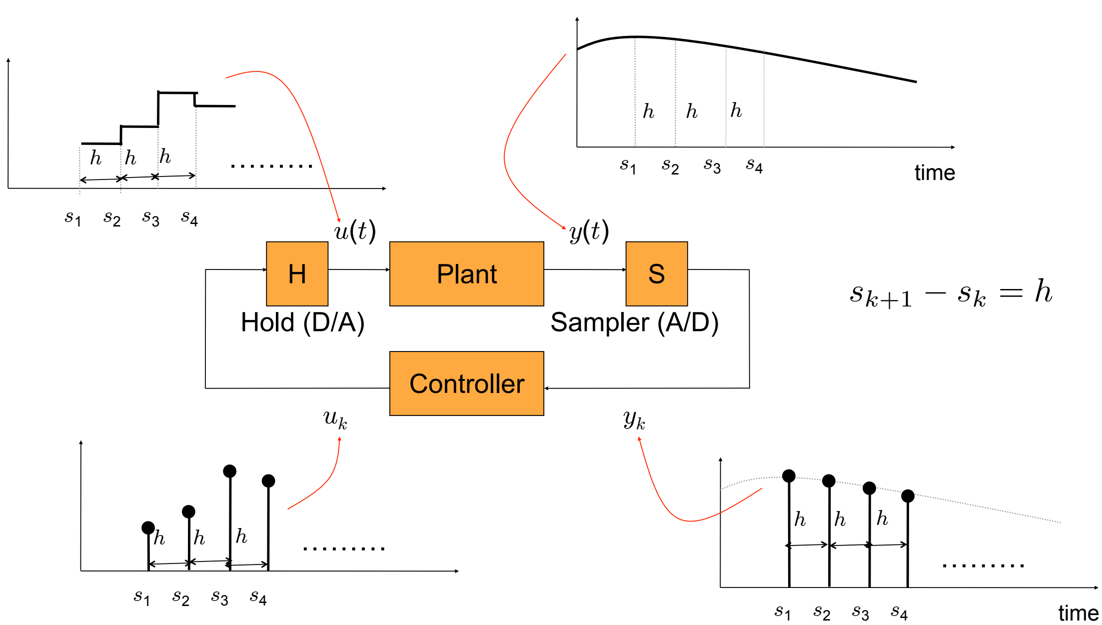
node	1 2 3 4 5 6 7
priorities 1	1 2 3 4 5 6 7
priorities 2	7 1 2 3 4 5 6

network access priorities

Ben Gaid, Cela,Kocik

Digital Control Systems

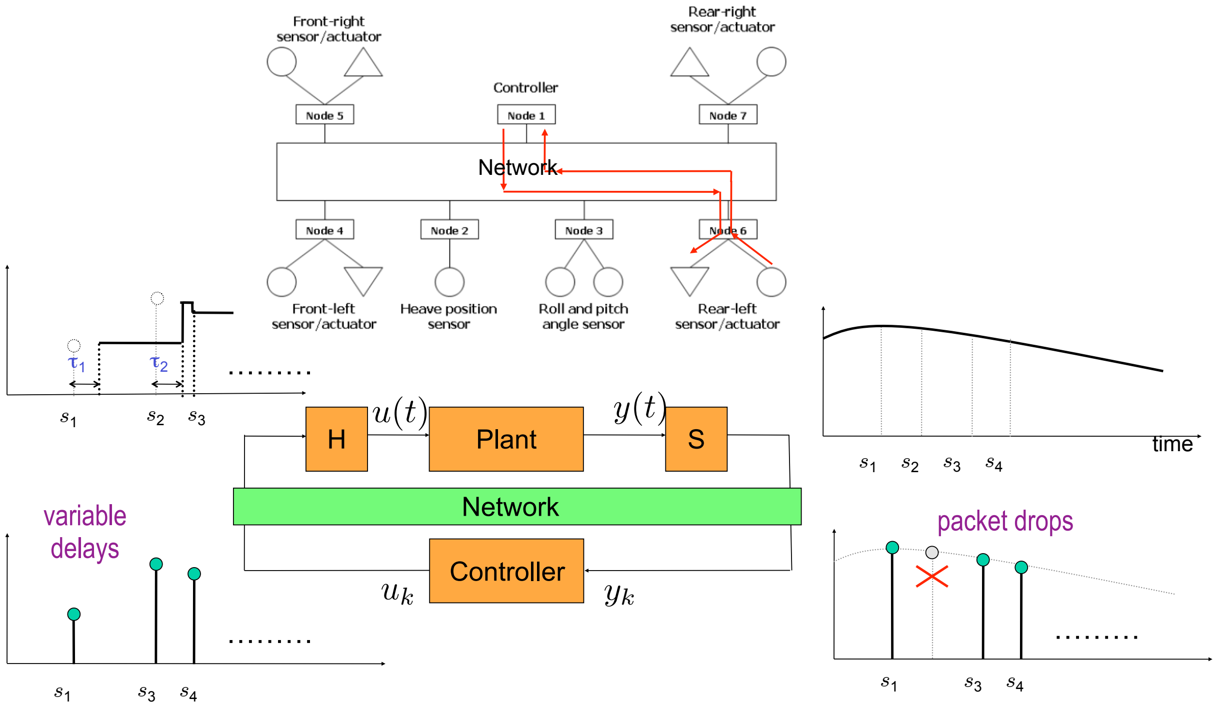
Digital control systems usually exhibit uniform sampling intervals and delays



$$s_{k+1} - s_k = h$$

Non-uniform Sampling/Delays

- Uniform sampling cannot be guaranteed (packet drops, clock synchronization, …)
- Different samples may experience different delays
- Difficult to decouple continuous plant from discrete events (sampling, drops, …)



Variable Delay Can Lead to Instability

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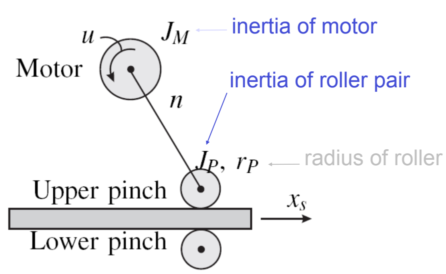
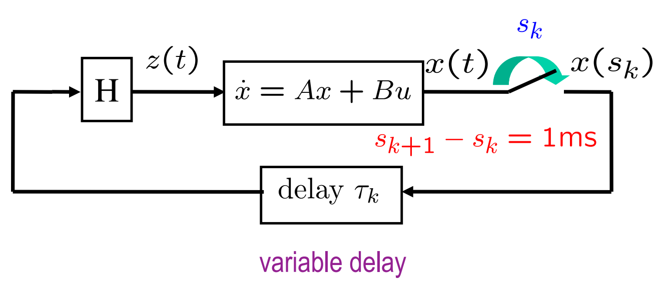
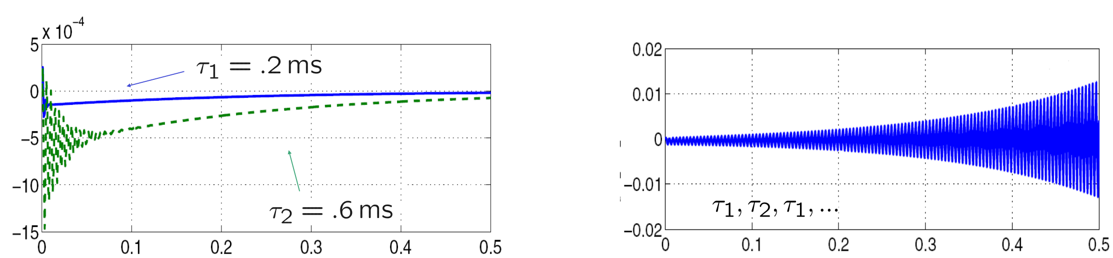




n: trans. ratio between motor and roller
x_s: sheet position
u: motor torque



Cloosterman and van de Wouw (Eindhoven University)

just variable sampling can lead to instability (even without drops)

Systems With Delays

Feedback loop with fixed delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx(t-\tau)$$

(fixed) delay in measuring $x(t)$

Feedback loop with variable delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx\big(t-\tau(t)\big)$$

time-varying delay

Classical Analysis

Feedback loop with fixed delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx(t-\tau)$$

time domain

time domain (Laplace transform)

$$sX(s) = (A + BKe^{-\tau s})X(s)$$

frequency domain

$$\text{poles of the system} \equiv \Big\{ s \in \mathbb{C} : \det\big(sI - (A + BKe^{-\tau s})\big) = 0 \Big\}$$

stability $\Leftrightarrow$ poles with negative real part (algebraic condition!)

Feedback loop with variable delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx\big(t-\tau(t)\big)$$

time-varying delay

frequency domain analysis does not lead to simple algebraic conditions!

Classical Analysis

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Feedback loop with fixed delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx(t-\tau)$$

time domain

$$sX(s) = (A + BKe^{-\tau s})X(s)$$

frequency domain

time domain
(Laplace transform)

poles of the system $= \left\{ s \in \mathbb{C} : \det\left(sI - (A + BKe^{-\tau s})\right) = 0 \right\}$

Lyapunov-based tools allow us to design controllers for NCSs that maintain performance under
- variable delays
- variable sampling rate
- network drops, etc.

Feedback loop with variable delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx\big(t - \tau(t)\big)$$

time-varying delay

frequency domain analysis does not lead to simple algebraic conditions!

Lyapunov-based Analysis

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Feedback loop with variable delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \quad u(t) = Kx\big(t - \tau(t)\big)$$

time-varying delay

Lyapunov-based analysis

$$V(x) := \|x\|^2 \qquad \frac{\mathrm{d}V(x)}{\mathrm{d}t} = \frac{\partial V(x)}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} \cdots < 0 \quad \Rightarrow \text{stability!}$$

- this “simplest” Lyapunov function is unlikely to “work,” but ...
- one can use numerical optimization techniques to find appropriate functions (actually functionals)
- stability conditions appear as feasibility problems that can be solved numerically very efficiently

- to apply these methods we need to find appropriate model for NCSs with delays…

Delay Impulsive Systems

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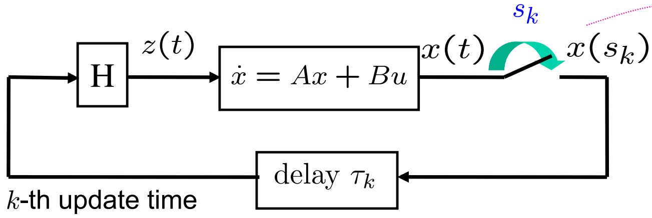
Single-channel NCS

k-th sampling time

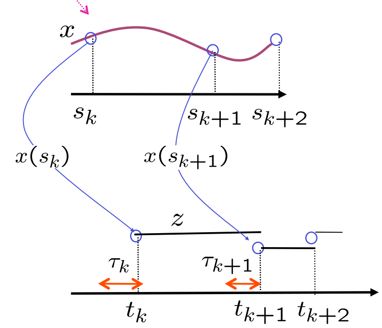
s_k



k-th update time

$t_k := s_k + \tau_k$

variable delay



deterministic delayed impulsive system (time driven)

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax + Bz \\ 0 \end{bmatrix}, \qquad t \neq t_k, \forall k \in \mathbb{N}$$

$$\begin{bmatrix} x(t_k) \\ z(t_k) \end{bmatrix} = \begin{bmatrix} x^-(t_k) \\ x(\underbrace{t_k - \tau_k}_{s_k}) \end{bmatrix}, \qquad t = t_k, \forall k \in \mathbb{N}$$

$$x^-(t) := \lim_{\tau \uparrow t} x(\tau)$$

Stability of Delay Impulsive Systems

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Consider delay impulsive system

$$\dot{x} = f_k(x,t), \qquad t \neq t_k, \forall k \in \mathbb{N},$$
$$x(t_{k+1}) = g_k(x^-(t_{k+1}), x(t_{k+1} - \tau_k)) \qquad t = t_k, \forall k \in \mathbb{N}.$$

System is GUES if there exists a Lyapunov functional

$$V : C\big([-r, 0], \mathbb{R}^n\big) \times \mathbb{R}^+ \to \mathbb{R}^+$$

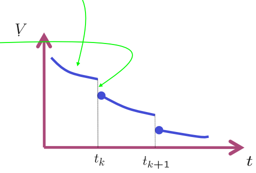
such that

state x truncated to the last r time units

(a) $d_1|\phi(0)|^b \leq V(\phi, t) \leq d_2|\phi(0)|^b + \bar{d}_2 \int_{t-r}^{t} |\phi(s)|^b ds \qquad \forall \phi \in C\big([-r, 0]\big), t \in \mathbb{R}^+$

(b) $\dfrac{dV(x_t, t)}{dt} \leq -d_3|x(t)|^b \qquad t \neq t_k, \ \forall k \in \mathbb{N}$

(c) $V(x_{t_k}, t_k) \leq \lim_{t \uparrow t_k} V(x_t, t) \qquad t = t_k, \forall k \in \mathbb{N}$



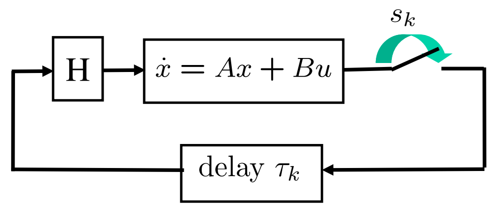
for $d_1, d_2, \bar{d}_2, d_3, b > 0$,

state x truncated to the last r time units

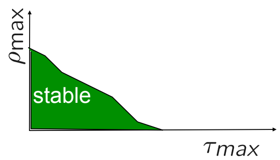
- Extended version of Lyapunov-Krasovskii Theorem for delayed systems with jumps.
- Lead to LMIs for linear systems

Stability of Single-Channel NCSs

Based on previous theorem and an appropriate choice of functional …



- There exists a set of pairs $(\rho_{\max}, \tau_{\max})$



such that

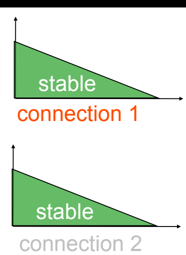
$$s_{k+1} - s_k \le \rho\text{max} \qquad \forall k \in \mathbb{N} \quad \Rightarrow \quad \text{exponential stability of the closed loop}$$
$$0 \le \tau_k \le \tau\text{max}$$

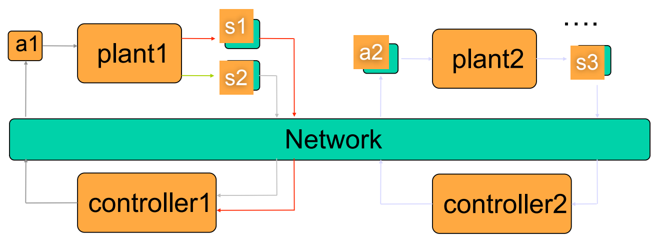
- We find the stability region by solving Linear Matrix Inequalities (LMIs)

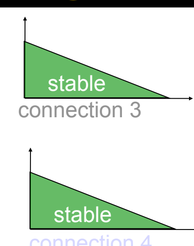
$$V := x'Px + \int_{t-\rho}^{t} (\rho\text{max} - t + s)\dot{x}'(s)R_1\dot{x}(s)ds + \int_{t-\sigma}^{t} (\sigma\text{max} - t + s)\dot{x}'(s)R_2\dot{x}(s)ds + \cdots$$

$$\rho(t) := t - s_k,\ \sigma(t) := t - t_k \quad t_k \le t < t_{k+1} \cdots$$

Stability of Multi-Channel NCSs

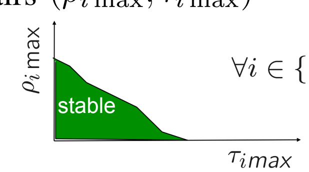






kth sampling time of channel i $\equiv s_k^i$
kth update time of channel i $\equiv t_k^i := s_k^i + \tau_k^i$

- There exists a set of pairs $(\rho_{i\,\max}, \tau_{i\,\max})$ $\quad \forall i \in \{1, \cdots, m\}$



such that

$$s_{k+1}^i - s_k^i \le \rho_{i\,\max},$$

delay = $\tau_k^i = b_k^i + C_k^i \le \tau_i\text{max}$ $\quad \Rightarrow$ exponential stability of all closed loops

blocking delay

transmission + propagation delay

Stability of Multi-Channel NCSs

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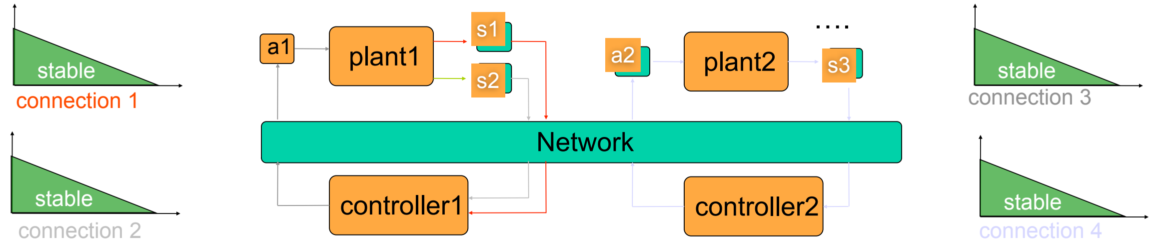
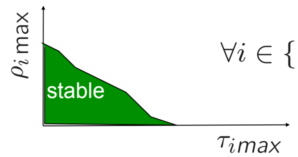


k*th* sampling time of channel $i \equiv s_k^i$

k*th* update time of channel $i \equiv t_k^i := s_k^i + \tau_k^i$

- There exists a set of pairs $(\rho_{i\,\max}, \tau_{i\,\max})$



$\forall i \in \{1, \cdots, m\}$

such that

$s_{k+1}^i - s_k^i \le \rho_{i\,\max}$,

delay = $\tau_k^i = b_k^i + C_k^i \le \tau_{i\max}$

$\Rightarrow$ exponential stability of all closed loops

blocking delay

transmission + propagation delay

- These inequalities define **deadlines for transmission delivery** (to be used, e.g., by Earliest Deadline First – EDF – scheduling)
- Blocking delay depends on **priority assignment**

Stable EDF scheduling

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Suppose:

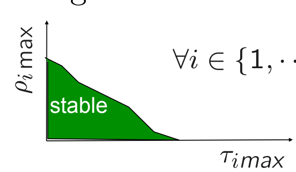
do not sample too fast

- $\rho_{i\,\min} \le s_{k+1}^i - s_k^i \le \rho_{i\,\max}, \forall k \in \mathbb{N}, i \in \{1, \cdots, n\}$
- $\sum_{i=1}^{n} \frac{C_i}{\rho_{i\,\min}} \le 1$

fastest sample does not exceed capacity

- $\sum_{i=1}^{n} \lfloor \frac{t - \tau_{i\,\max}}{\rho_{i\,\min}} \rfloor^+ C_i + \max_i C_i \le t \quad \forall t \in S$

and every pair $(\rho_{i\,\max}, \tau_{i\,\max})$ belongs to the shaded region



$\forall i \in \{1, \cdots, m\}$

can be implemented, e.g., using CAN priorities

Then the following holds for EDF scheduling

$\tau_k^i = b_k^i + C_k^i \le \tau_{i\max} \quad \Rightarrow$ exponential stability of all closed loops

$$S = \bigcup_{i=1}^{n} \{\tau_{i\,\max} + h\rho_{i\,\min}, h = 0, 1, \cdots, \overline{\lfloor \frac{d - \tau_{i\,\max}}{\rho_{i\,\min}} \rfloor}\}, d := \ldots \quad \lfloor x \rfloor^+ := \ldots$$

Example: motion control system for sheet control

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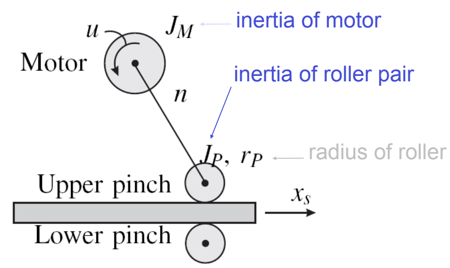
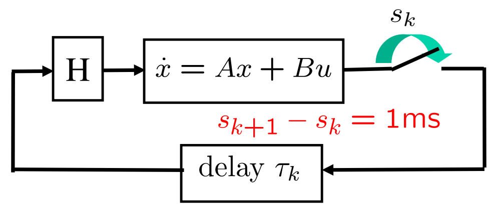


n: trans. ratio between motor and roller
x_s: sheet position
u: motor torque



$$x = \begin{bmatrix} x_s \\ \dot{x}_s \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = -\begin{bmatrix} 0 \\ b \end{bmatrix} \times \begin{bmatrix} 50 & 11.8 \end{bmatrix}$$

$$b := \frac{n r_R}{J_M + n^2 J_R}$$

Controller gain

- Position and velocity measurements are sent to an ECU through a CAN network
- ECU computes control commands and applies to motors directly, which takes *0.1ms*
- Transmission time is C_i= *1 ms* (*8* bytes, *64* kbit/s)
- Closed-loop system remains stable for any constant sampling smaller than *48 ms* when delay=0

⇒ we choose sampling interval =*12 ms*

Example (continued)

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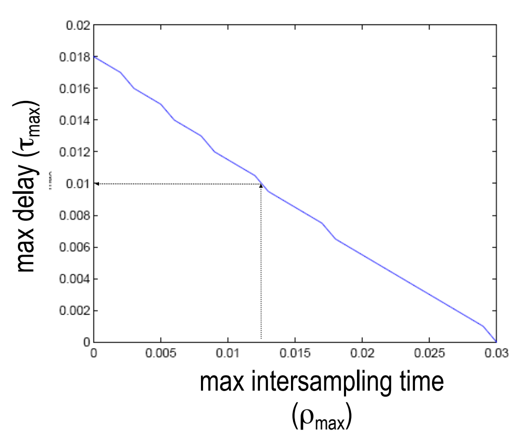
How many motors can be controlled?

- Ad-hoc approach: a conservative designer *n=6* so bus load 50%
  an aggressive designer *n=11* so bus load just below 100% (91.7%)

Our approach:

- By solving the LMIs we find admissible set of sampling-delays. For sampling=*12 ms*, max variable delay=*10ms*
- By testing scheduling condition with T_i=*12ms*, D_i=*10-0.1=9.9ms*, C_i=*1ms* we conclude *n=9* (bus load *75%)*
- By following the proposed method we avoid conservative choices and 'unsafe' choices



Network protocols & Control laws

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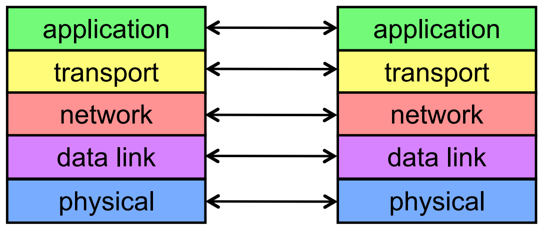
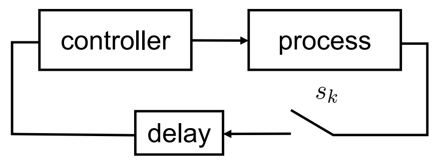
network view:



control view:



This lecture: ***Co-design of network protocols and control algorithms***

1. Characterize *key parameters* that determine the stability/performance of a networked controls system
2. Construct *protocols* that directly take these parameters into considerations

Illustrative examples:

- data link layer: medium access control
- transport layer: error correction (& flow control)
- network layer: routing

Transport layer protocols

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Most common (general purpose) protocols:

high drop rates can lead to poor performance and eventually instability

UDP

- no attempt at error correction
- no attempt to control data rate

TCP

- error correction
  - all packets sent should be acknowledged by receiver
  - lack of acknowledgement of packet n leads to retransmission of same packet after packet $n + 3$ (triple duplicate ack mechanism)
- congestion control
  - packet drops are taken as a sign of con … rease

delayed retransmissions are essentially useless; too much overhead in ack every packet

Illustrative 1-D problem

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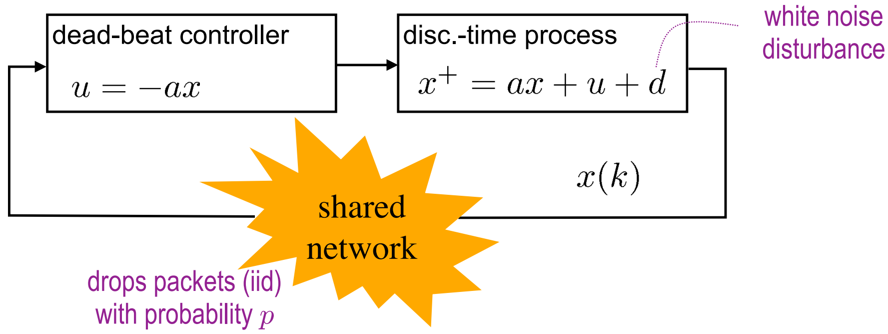

The closed-loop is *mean-square stable* (i.e., E[$x(k)^2$] <∞) if and only if

$$p < \frac{1}{|a|^2}$$

(it is also straightforward to compute a tight asymptotic bound on E[$x(k)^2$])

Intuition: ignoring the disturbance d

$$x(k+1)^2 = \begin{cases} 0 & \text{with probability } 1-p \\ |a|^2 x(k)^2 & \text{with probability } p \end{cases} \quad \Rightarrow \quad \mathrm{E}[x(k+1)^2] = p\,|a|^2\,\mathrm{E}[x(k)^2]$$

Illustrative 1-D problem

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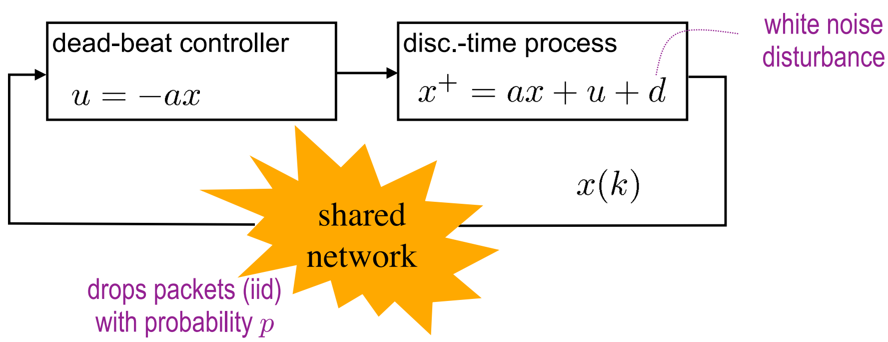


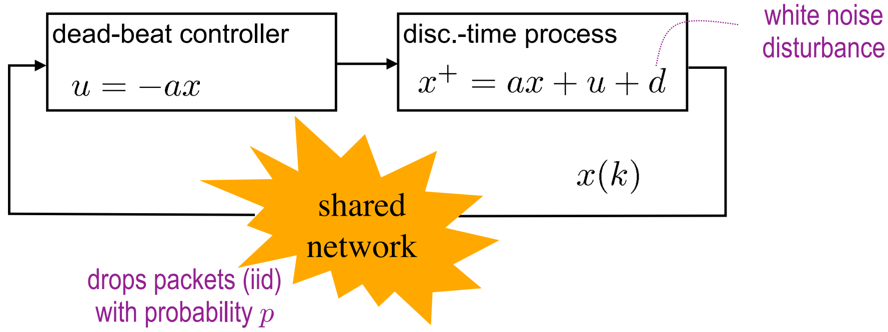
The closed-loop is *mean-square stable* (i.e., E[$x(k)^2$] <∞) if and only if

$$p < \frac{1}{|a|^2}$$

(it is also straightforward to compute a tight asymptotic bound on E[$x(k)^2$])

But what if $|a|>1$ *and the probability of drop is larger than this bound?*

Redundant transmissions



redundant transmissions ≡ at each time step one sends N copies of $x(k)$ through independent channels (time, frequency, or spatial diversity), each with drop probability p

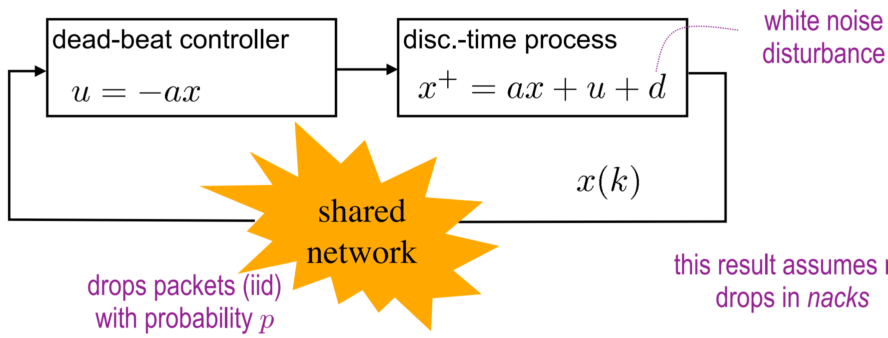
The closed-loop is *mean-square stable* (i.e., E[$x(k)^2$] <∞) if and only if

$$p^N < \frac{1}{|a|^2} \quad \Leftrightarrow \quad p < \frac{1}{|a|^{\frac{2}{N}}}$$

any drop probability can be accommodated by choosing N sufficiently large

but transmission rate is *N times larger*

A simple "error-correction" protocol



this result assumes no drops in *nacks*

1. when a packet is lost, receiver sends a "negative acknowledgement" (*nack*)
2. transmitter generally sends *one* packet at each sampling time, however…
3. upon reception of *nack*, transmitter sends *two* copies of the following packet

The closed-loop is *mean-square stable* (i.e., E[$x(k)^2$] <∞) if and only if

$$p < \frac{1}{|a|}$$

similar bound as if always sending two packets

but average transmission rate is only *1+O(p) times larger*

[ACC'09]

Even better…

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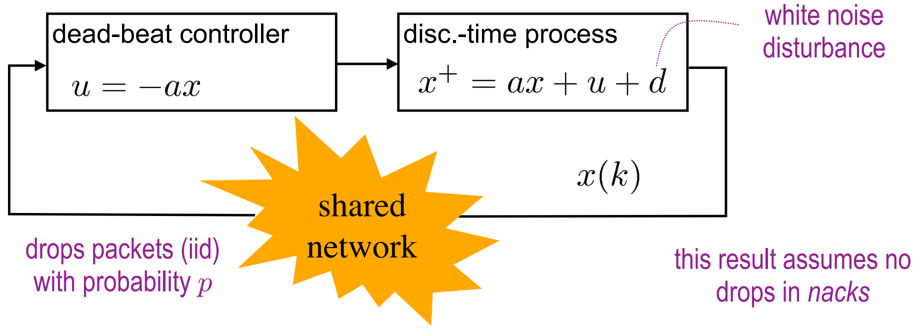

Pick a function $v : \mathbb{N} \to \mathbb{N}$, with $v(0) = 1$

1. when a packet is lost, receiver sends a "negative acknowledgement" (*nack*)
2. transmitter keeps track of number $\ell(k)$ of consecutive drops prior to time k

- transmitter sends $v(\ell(k))$ copies of each packet

For every p, a, and N, one can find a function $v : \mathbb{N} \to \mathbb{N}$ such that

- closed-loop is *mean-square stable* (i.e., E[$x(k)^2$] < ∞) — stabilizes any system
- average transmission rate is only *1+O(p^N) times larger* — arbitrarily small increase in the transmission rate
- requires at least N independent channels — all but one channel are rarely utilized

Even better…

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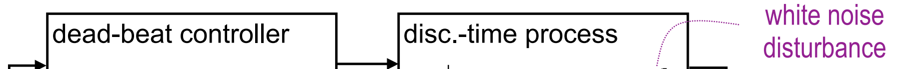
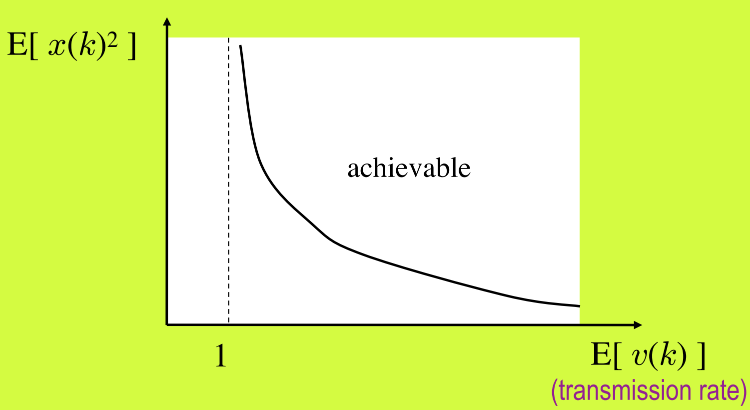


- can stabilize any system for any drop probability
- with arbitrarily small increase in the transmission rate

no (completely) free lunch… E[$x(k)^2$] will be large



ult assumes no ps in *nacks*

Pick a fu

1.when a … *nack*)
2.transm … me k

- transm

For every

- closed … ny system
- average transmission rate is only *1+O(p^N) times larger* — arbitrarily small increase in the transmission rate
- requires at least N independent channels — all but one channel are rarely utilized

Optimal "error-correction" protocols

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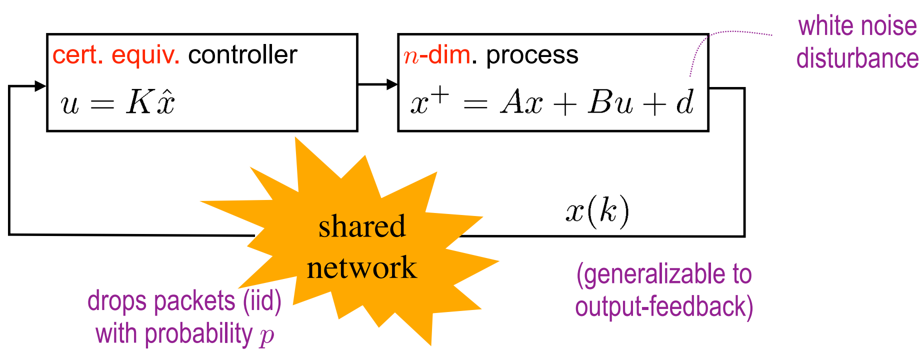

choose $v(k) \equiv$ number of copies of $x(k)$ to send at time instant k

to minimize

$$\lim_{N\to\infty} \underbrace{\left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1} \|x(k)-\hat{x}(k)\|^2\right]\right)}_{\text{state estimation error (performance)}} + \lambda \underbrace{\left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1} v(k)\right]\right)}_{\text{transmission rate (communication)}}$$

average-cost optimal control of a Markov process on $\mathbb{R}^n$

Optimal "error-correction" protocols

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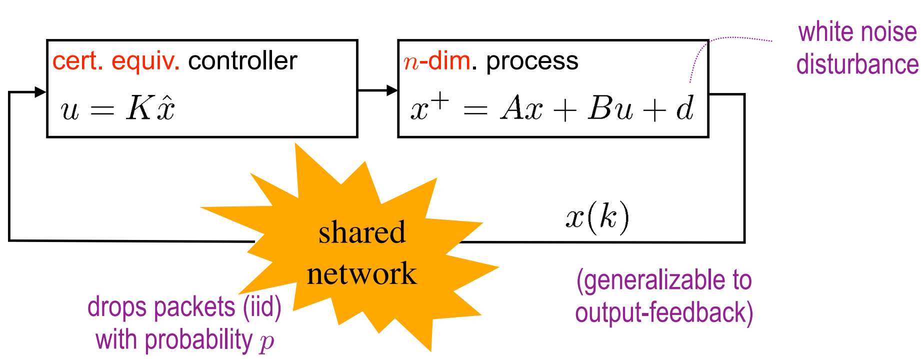


$$\lim_{N\to\infty} \left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1} \|x(k)-\hat{x}(k)\|^2\right]\right) + \lambda \left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1} v(k)\right]\right)$$

Theorem:

- optimal $v(k)$ is generated by a memoryless policy of the form

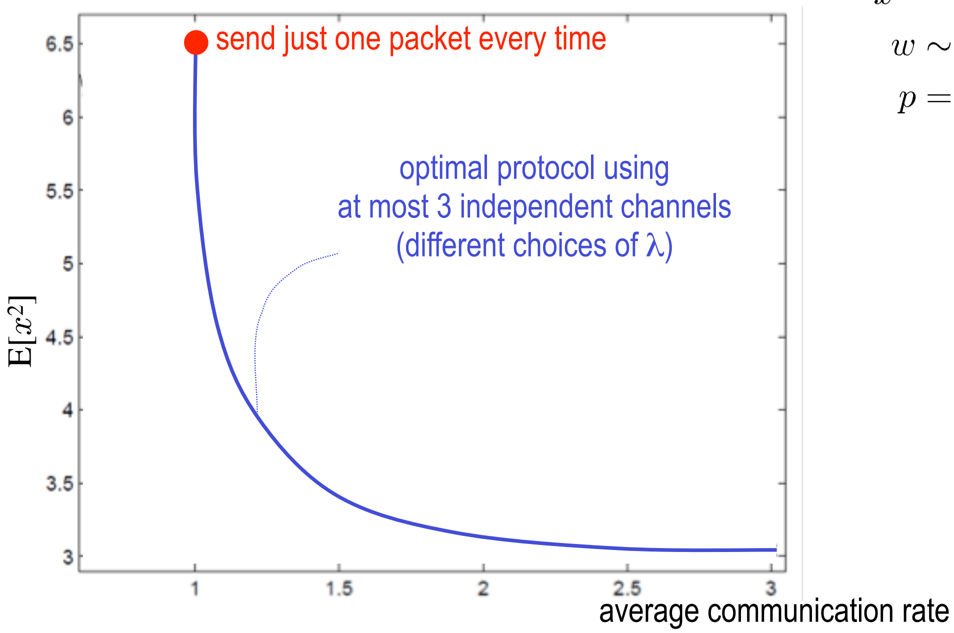
$$v(k) = \pi^*\big(x(k) - \hat{x}(k)\big)$$

transmitter must construct a state estimate to determine optimal $v(k)$

- optimal policy π^* can be computed using dynamic programming and value-iteration

computationally difficult for large n

Example



$$x^+ = 2x + u + w$$
$$w \sim N(0,3)$$
$$p = .15$$

Optimal "simplified" protocols

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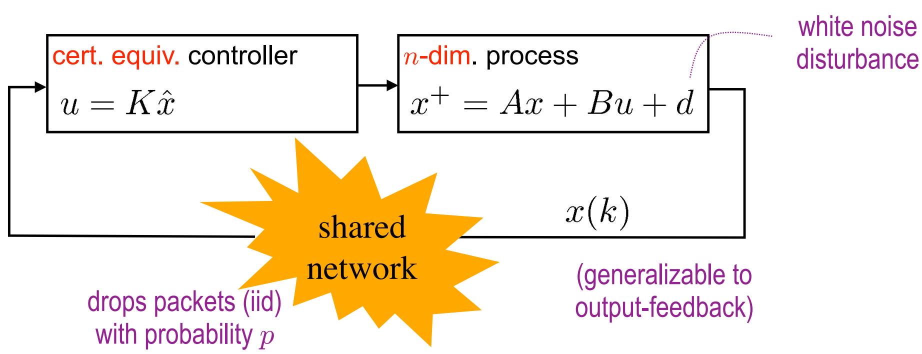


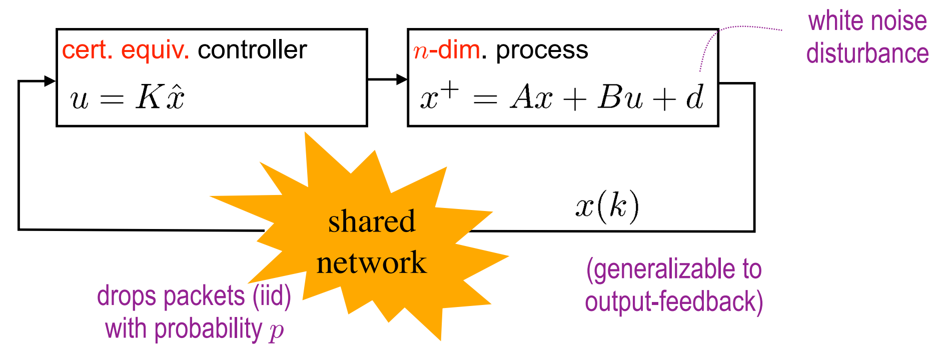
choose $v(k) \equiv$ number of copies of $x(k)$ to send at time instant k

to minimize

$$\lim_{N\to\infty} \underbrace{\left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1}\|x(k)-\hat{x}(k)\|^2\right]\right)}_{\substack{\text{state estimation error}\\ \text{(performance)}}} + \lambda \underbrace{\left(\frac{1}{N}\,\mathrm{E}\left[\sum_{k=0}^{N-1} v(k)\right]\right)}_{\substack{\text{transmission rate}\\ \text{(communication)}}}$$

but transmitter must choose $v(k)$ based only on # of consecutive drops (from nacks)

Optimal “simplified” protocols



$$\lim_{N\to\infty} \left(\frac{1}{N} \mathrm{E} \left[\sum_{k=0}^{N-1} \|x(k) - \hat{x}(k)\|^2 \right] \right) + \lambda \left(\frac{1}{N} \mathrm{E} \left[\sum_{k=0}^{N-1} v(k) \right] \right)$$

Theorem:

- optimal $v(k)$ is generated by a memoryless policy of the form

$$v(k) = \pi^*\big(\ell(k)\big)$$

transmitter only needs to keep track of
$\ell(k) \equiv$ # of consecutive drops (from nacks)

- optimal policy π^* can be computed using dynamic programming and value-iteration

computationally much easier
(optimization on countable-state
MDP with size independent of n)

Example

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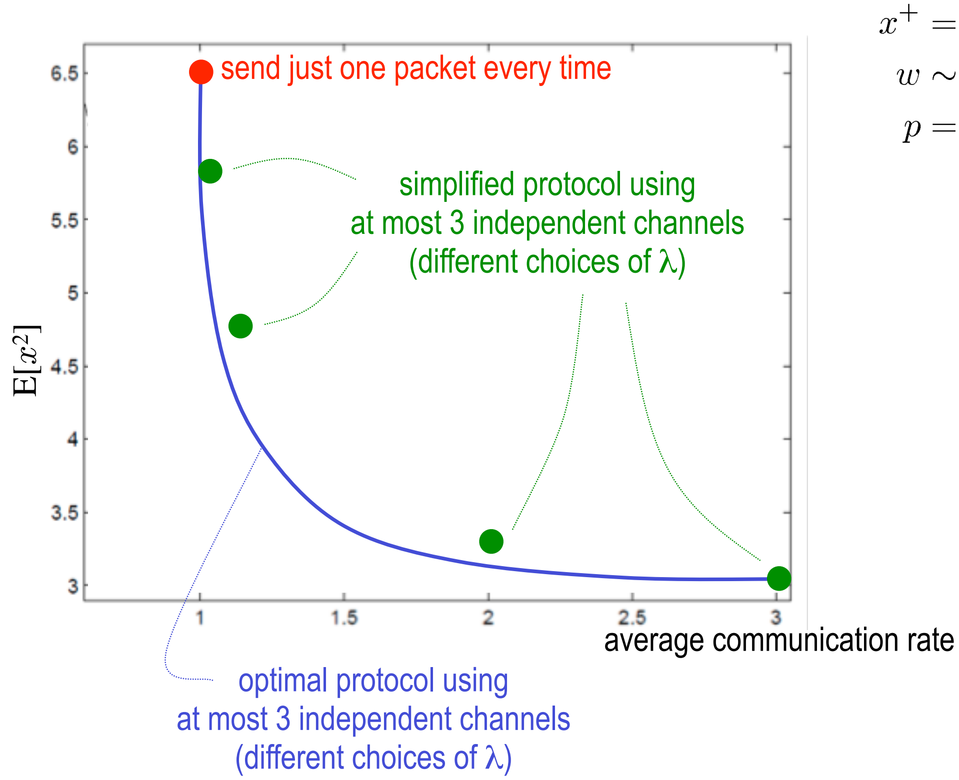
$$x^+ = 2x + u + w$$
$$w \sim N(0,3)$$
$$p = .15$$



Example

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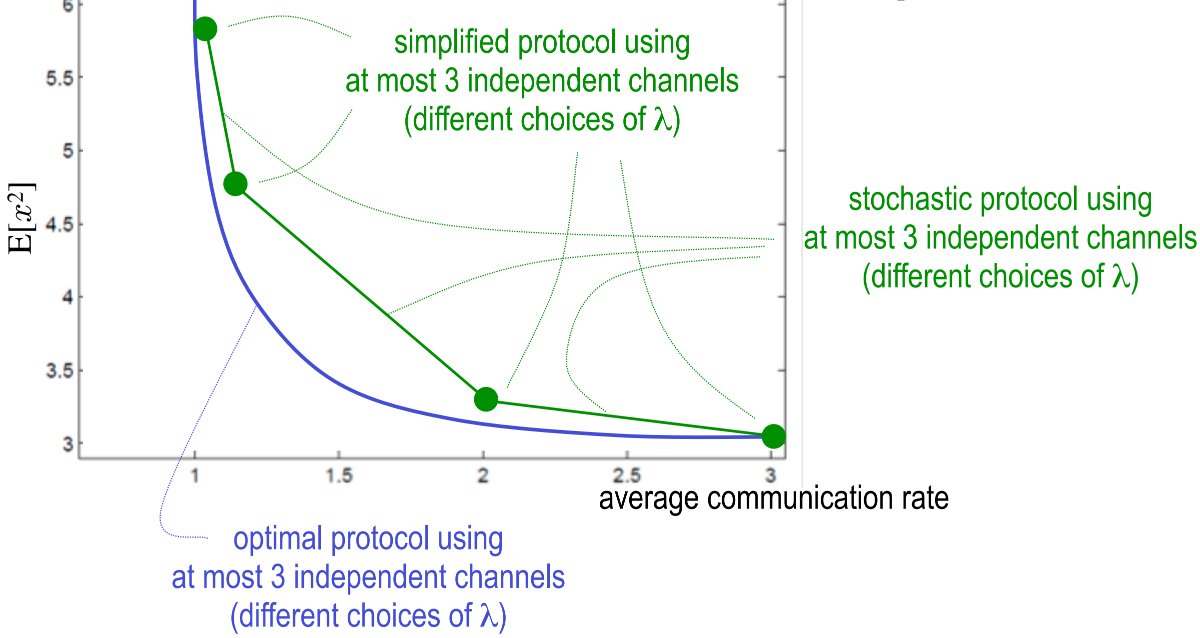
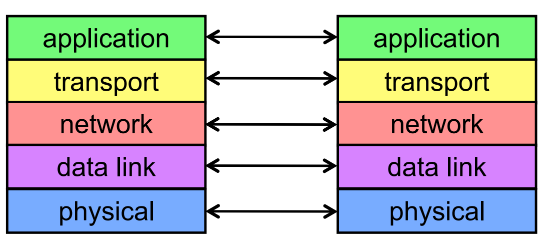
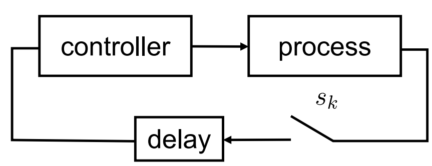


Network protocols & Control laws

network view:



control view:



This lecture: ***Co-design of network protocols and control algorithms***

1. Characterize *key parameters* that determine the stability/performance of a networked controls system
2. Construct *protocols* that directly take these parameters into considerations

Illustrative examples:

- data link layer: medium access control
- transport layer: error correction (& flow control)
- network layer: routing

Problem formulation

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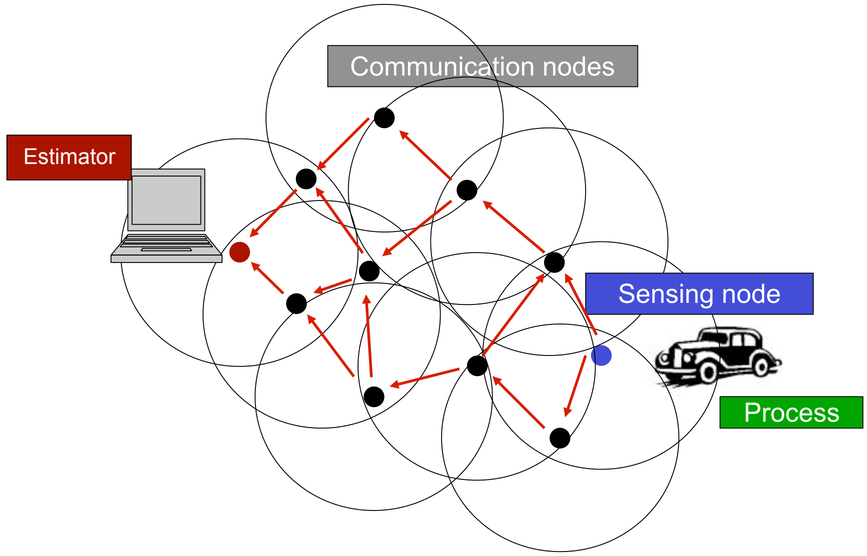

Estimation of a process across a network

Multi-hop/multi-path communication between sensor and estimator

Communication Channel Model

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$$\begin{bmatrix} 1 \\ 2.3 \\ 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 2.3 \\ 6 \end{bmatrix}$$

Received without error

Packet Erasure Model

$$\begin{bmatrix} 1 \\ 2.3 \\ 6 \end{bmatrix} \longrightarrow \begin{bmatrix} \phi \\ \phi \\ \phi \end{bmatrix}$$

Packet dropped

1. Enough quantization bits
2. No corruption of data

System Model

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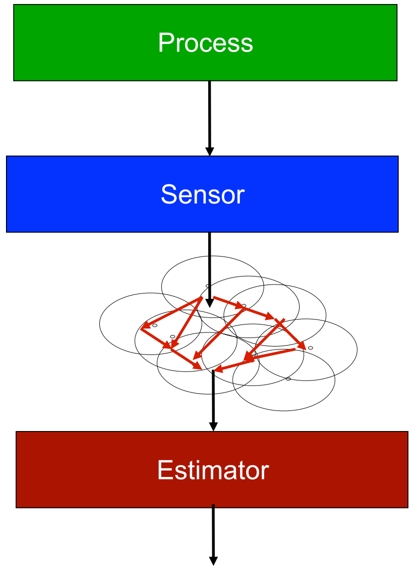

$$x_{k+1} = Ax_k + w_k$$

$$y_k = Cx_k + v_k$$

- Packet Erasure Channels
- Network with Arbitrary Topology

Minimum Mean Squared Error Estimator

Minimize, *at every time step*, the mean squared cost $E\left[(x_k - \hat{x}_k)^T (x_k - \hat{x}_k)\right]$

The Network Case

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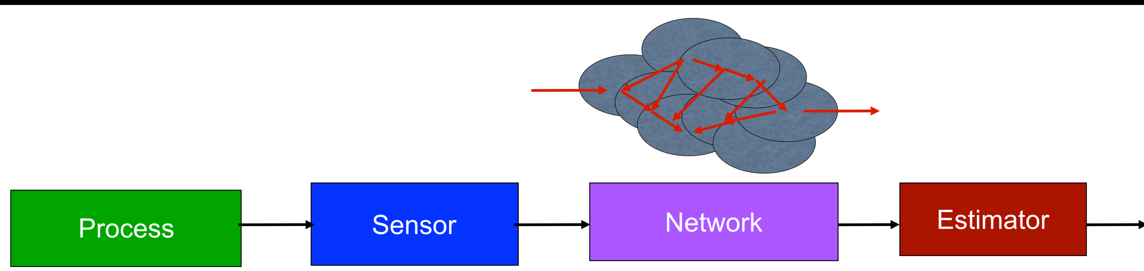


If the sensor (and every intermediate node) simply transmits measurements, the network is equivalent to a single channel with

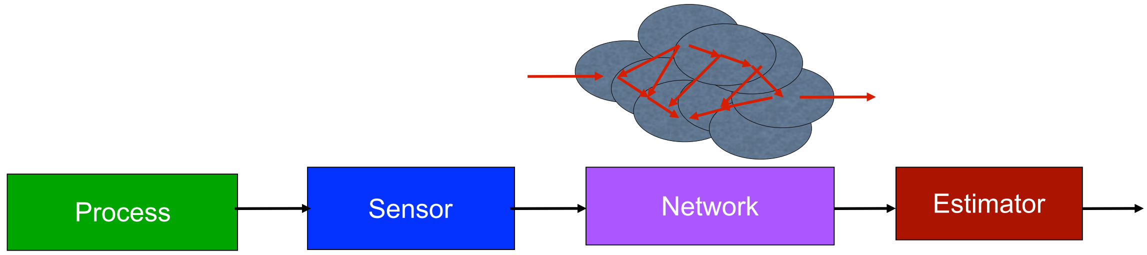
equivalent drop probability = 1 – "reliability of the network"

For a *series combination* of n links each with drop probability p, the 'equivalent' drop probability is $$1-(1-p)^n \approx np \; (p \ll 1)$$

For $n = 5, p = 5\%$, the 'equivalent' drop probability is 23%.

Can we do better ?

In This Lecture



- Theme: Use (limited) memory and processing power at the intermediate nodes to obtain better performance.

- If the nodes follow a recursive algorithm, optimal performance is achieved.
- Stability conditions can be checked simply.

Is it Feasible?

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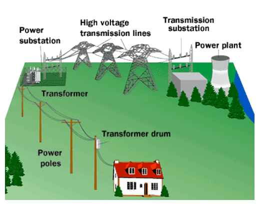
Telos wireless network modules from Moteiv

1. Microcontroller: 8 MHz Texas Instrument MSP430
2. Program memory: 62K
3. Flash memory: 256K

MVWT-II vehicles at Caltech

1. Microcontroller: 206 MHz Zaurus PDA
2. Flash memory: 16M

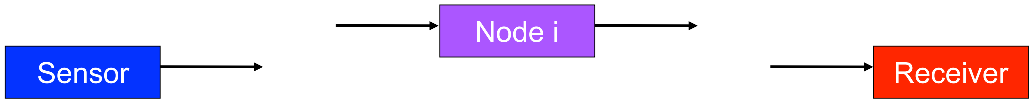


Power grid

Ample processing and memory capabilities

Constraint: memory and computation required should not increase with time.

Information Processing Algorithm



Every node keeps

1. an estimate of current state value based on all data it has received so far &
2. a time-stamp of the latest measurement used to construct this estimate.

Kalman filter at the sensor.

Switched linear filter at all other nodes.

- Compare the time-stamps of the estimates received along incoming edges with the one in memory.
- Choose the estimate with the most recent time-stamp.
- Update estimate and transmit it along outgoing edges.

Properties of the Algorithm

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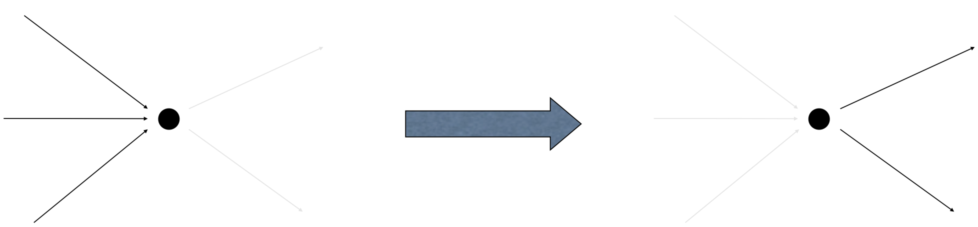

Choose the estimate that uses latest measurement

Update and transmit it along outgoing edges

- Same performance as transmitting all previous measurements at every time step (“optimality”).
- Constant amount of transmission and memory required.
- Each received packet ‘washes away’ the effect of all previous drops.

Optimal for arbitrary network (may even have cycles).
No assumption needed about the packet dropping process.

Special Networks: Stability of Error Covariance

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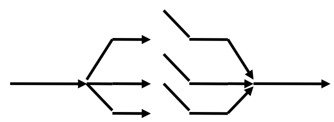
$$x_{k+1} = Ax_k + w_k \qquad \text{(process)}$$

Necessary and sufficient conditions for boundedness of the error covariance (mean-square stability)

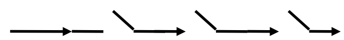
Parallel Networks:



$$\left(\prod_i p_i\right) \mid \rho(A) \mid^2 < 1$$

(independent drops assumed for simplicity)

Series Networks:



$$\left(\max_i p_i\right) \mid \rho(A) \mid^2 < 1$$

For a series combination of n links each with same drop probability p, the condition is

$$p \mid \rho(A) \mid^2 < 1$$

But if transmitting measurements it would be $\quad (1 - (1-p)^n) \mid \rho(A) \mid^2 < 1$

$$\hookrightarrow \; np \mid \rho(A) \mid^2 \lessapprox 1 \; (p \ll 1)$$

General Networks: Stability of Error Covariance

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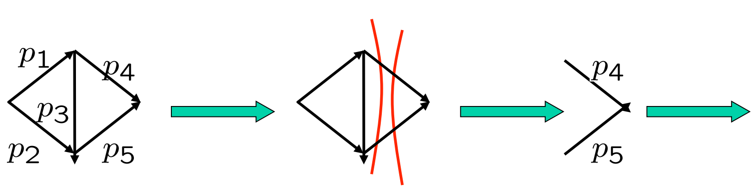
Necessary and sufficient conditions for boundedness of the error covariance (mean-square stability)

General networks:

$$p_{max-cut} \mid \rho(A) \mid^2 < 1$$

Max-cut probability

1. For each cut-set, identify edges that span from the source set to the sink set.
2. Calculate the cut-set probability: $\quad p_{cut} = \left(\prod_i p_i\right)$
3. Identify the maximum cut-set probability $p_{max-cut}$.



Cut-set probability = $p_4 p_5$

General Networks: Performance

The expected steady state error covariance can be evaluated using a closed formula

process matrix   ideal cov.

$$\text{vec}\left(P^{\infty}\right) = \left(\left(A \otimes A - I\right) G\left(A \otimes A\right) + I\right) \text{vec}\left(P^{\star}\right) + G\left(A \otimes A\right) \text{vec}\left(Q\right)$$

network   noise cov.

Details in Dana et al. (TAC)

Summary

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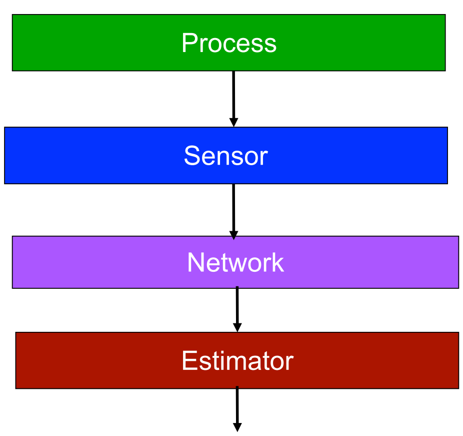
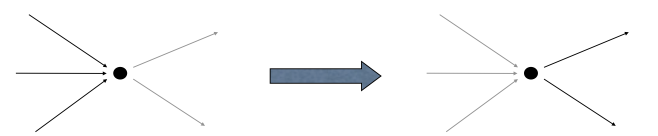




Choose the estimate that uses latest measurement

Time update and transmit along outgoing edges

Condition for Stability of Error Covariance

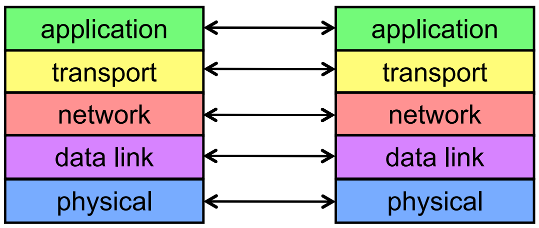
$$p_{max-cut} \mid \rho(A) \mid^2 < 1$$

Use (limited) memory and processing power at the intermediate nodes to obtain better performance.

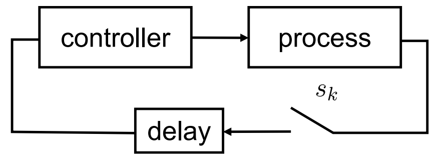
- **Recursive algorithm for optimal performance identified.**
- **Necessary and sufficient stability conditions provided.**

Network protocols & Control laws

network view:



control view:



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END